ORIE 6300 Mathematical Programming I	September 20, 2016

Lecture 9

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1 Last Time

- We introduced value functions: $V(u) = \max\{c^T x | Ax \leq b + u\}$. We found that the supgradients of a value function are exactly the dual optimal solutions.
- Following from our analysis of value functions, we found that we can always perturb a linear program slightly to make the optimal solution unique.
- We proved Fourier-Motzkin's Theorem, which states that $LQ(A, b)$ is a polyhedron for any linear transformation L. As a consequence of this, we find that for any two polyhedral sets S_1, S_2 the sets $S_1 + S_2$ and $S_1 - S_2$ are closed. This fact was necessary in lectures 7/8 to show the existence of minimizers and maximizers. We also need this result to show $\{A^T y | y \geq 0, y^T(b - Ax) = 0\}$ is closed.

2 Verifying Optimality

Consider a primal and dual LP in the generic form in which we have been studying LPs so far in the course, in the case when both are feasible. We know the optimal values of the LPs are equal, but is there a good procedure to tell whether a given $\bar{x}$ is optimal? Let's look at the following LP primal and dual pair:

$$\begin{array}{llcll} \min & c^T x & \qquad & \max & y^T b \\ \text{s.t.} & Ax \leq b & & \text{s.t.} & A^T y = c \\ & & & & y \geq 0 \end{array} \tag{1}$$

Given primal and dual feasible $\bar{x}$ and $\bar{y}$, how do we determine that they are optimal? From the strong duality theorem, we know if $\bar{x}$ and $\bar{y}$ are optimal, then $c^T \bar{x} = b^T \bar{y}$. For this to be true, we need that if $\sum_{i=1}^m a_{ij} \bar{x}_j < b_i$ then $\bar{y}_i = 0$. Call these conditions $(*)$.

Definition 1 *We say that a primal feasible solution $\bar{x}$, and a dual feasible solution $\bar{y}$ obey the* complementary slackness conditions *if $(*)$ holds.*

So we see from the above that if $\bar{x}$ and $\bar{y}$ are optimal solutions, then complementary slackness holds. But actually we can say something stronger than this.

Lemma 1 *Let $\bar{x}$ satisfy $A\bar{x} \leq b$. Then $\bar{x}$ is primal optimal if and only if there exists a $\bar{y} \geq 0$ such that $A^T \bar{y} = c$ and $\bar{y}^T(b - A\bar{x})$=0.*

Proof: We know $\bar{x}$ is optimal if and only if $c \in N_{Q(A,b)} = \{A^T y | y \geq 0, y^T(b - A\bar{x})\}$. Selecting such a y from the normal cone of $Q(A, b)$ suffices to complete the proof. □

Lemma 2 *Let $\bar{y}$ satisfy $A^T\bar{x} = c$ and $y \geq 0$. Then $\bar{y}$ is dual optimal if and only if there exists a $\bar{x}$ such that $A\bar{x} \leq b$ and $\bar{y}^T(b - A\bar{x})$=0.*

Proof: Left as an exercise. This will be similar to the proof of the previous Lemma. □

Hence we have an answer to our question. We find that $\bar{x}$ is primal optimal if there exists a dual feasible $\bar{y}$ such that the complementary slackness conditions hold. Further, $\bar{y}$ is dual optimal if there exists a primal feasible $\bar{x}$ such that complementary slackness holds.

This still doesn't seem like such a useful way of verifying optimality, but it will prove to be a step in the right direction. As we have done before, we can partition the columns of A into $A_= = A_=(\bar{x})$ and $A_< = A_<(\bar{x})$ (and similarly $b_=$ and $b_<$). We know that the rank of $A_=$ is n for any vertex $\bar{x}$. Then we can partition any dual solution y such that $A^Ty = c$ into $(y_=, y_<)^T$ to match the corresponding rows of A.

Now suppose y satisfies complementary slackness. We know that $b_< - A_<x > 0$, which immediately implies that $y_< = 0$. So we find the following:

$$A_=^T y + A_<^T y = c \implies A_=^T y = c$$

In the best case scenario, we would have exactly n rows in $A_=$. Then the matrix $A_=^T$ would be invertible. So we could compute $y_= = (A_=^T)^{-1}c$. If we find that $y_= \geq 0$, then y must be dual optimal, and thus our original $\bar{x}$ is primal optimal.

At this point, we will change our notation to have a standard form linear program be the primal problem. So our pair of primal and dual programs becomes the following:

$$\begin{array}{rlcrl} \min & c^T x & \qquad & \max & y^T b \\ \text{s.t.} & Ax = b & & \text{s.t.} & A^T y \leq c \\ & x \geq 0 & & & \end{array} \tag{2}$$

Notation 1 *We let $P(A, b) = \{x \mid Ax = b, x \geq 0\}$.*

So far we haven't been taking advantage of something that we know about optimal solutions. We know that there exists an optimal solution that is a vertex for standard form programs. Further, we have shown this on a problem set for bounded polyhedra, and in a recitation for pointed polyhedra. We've also shown in a problem set that if x is not a vertex, we can find a vertex $\tilde{x}$ such that $c^T\tilde{x} \leq c^Tx$. So we can assume that x is a vertex.

Lemma 3 *A feasible solution $\bar{x} \in P(A, b)$ is a vertex if and only if the columns corresponding to its positive coordinates are linearly independent.*

Proof: Let $C = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}$. Then we will prove the two directions of the lemma separately.

First, suppose $\bar{x}$ is a vertex. Then $C_=$ has rank n. Without loss of generality, we order x such that $x_1, ..., x_k > 0$ and $x_{k+1}, ..., x_n = 0$. Then we can partition A and C as follows:

$$A = \begin{bmatrix} A_B & A_N \end{bmatrix},\ C = \begin{bmatrix} A_B & A_N \\ -A_B & -A_N \\ 0 & -I \end{bmatrix}.$$

We find that $C_= = \begin{bmatrix} A_B & A_N \\ -A_B & -A_N \\ 0 & -I_{n-k} \end{bmatrix}$ must have rank n.

Then it follows that $\begin{bmatrix} A_B & A_N \\ 0 & -I_{n-k} \end{bmatrix}$ must also have rank n. Applying simple row operations will give us that $n = \text{rank}\left(\begin{bmatrix} A_B & A_N \\ 0 & -I_{n-k} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} A_B & 0 \\ 0 & -I_{n-k} \end{bmatrix}\right) = \text{rank}(A_B) + n - k$. Rearranging this gives $\text{rank}(A_B) = k$, which completes the forward direction of our proof.

Now we show the reverse direction. Again, suppose $x_1, ..., x_k > 0$ and $x_{k+1}, ..., x_n = 0$. Then we partition $A = \begin{bmatrix} A_B & A_N \end{bmatrix}$. By our assumption, we have $\text{rank}(A_B) = k$. Just like our forward direction proof, we know that $C_=$ has the following form:

$$C_= = \begin{bmatrix} A_B & A_N \\ -A_B & -A_N \\ 0 & -I_{n-k} \end{bmatrix}$$

Then $\text{rank}(C_=) = \text{rank}\left(\begin{bmatrix} A_B & A_N \\ 0 & -I_{n-k} \end{bmatrix}\right) = \text{rank}(A_B) + n - k = n$, completing our proof. $\square$

This gives us an easy way to check if a feasible solution is a vertex or not. It's worth encoding this into a definition. First, we need an assumption though. We assume without loss of generality that the m rows of A are linearly independent. It's without loss of generality since otherwise a constraint is redundant (if a constraint can be expressed as a linear combination of other constraints) or the system $Ax = b$ is infeasible.

Definition 2 *A set B of m columns of A is a* basis *if these columns are linearly independent.*

We will focus on a subset of columns of A which correspond to a basis B.

$$A\text{: } m \text{ lin. ind. rows} \quad \begin{array}{c} \begin{bmatrix} A & \begin{pmatrix} A_i \end{pmatrix} & \quad \end{bmatrix} \\ \uparrow \; \uparrow \; \uparrow \\ m \text{ columns } B \end{array} \quad \rightarrow \quad A_B$$

Note that for any basis B, the matrix A_B will be invertible.

We will denote by x_B the coordinates of x corresponding to basis B. We do the same for the *nonbasic* variables N, which correspond to all the columns of A not in B, and define A_N and x_N similarly. In the basic solution corresponding to basis B, we set the nonbasic variables to zero, so that $A_N x_N = 0$.

Definition 3 *We say x is a basic solution if $x_i \neq 0$ implies $i \in B$, for some basis B of A.*

Lemma 4 *For any basis B, there is a unique corresponding basic solution to $Ax = b$.*

Proof: To see this, notice that any such solution has to satisfy

$$\begin{bmatrix} A_B | A_N \end{bmatrix} \begin{bmatrix} x_B \\ - \\ x_N \end{bmatrix} = b$$

Notice that $A_N x_N = 0$, $Ax = b \;\Rightarrow\; A_B x_B + A_N x_N = b \;\Rightarrow\; A_B x_B = b$. Since A_B is an $m \times m$ matrix of rank m, the solution $x_B = A_B^{-1} b$ is uniquely determined. $\square$

However, the reverse of this lemma is not the case. There could be multiple basis associated with the same basic solution. To see this, suppose $x_B = A_B^{-1} b$ has some $i \in B$ such that $x_i = 0$. This could occur if $b = 0$, or in a much more general context.

Proposition 5 *Every vertex x of $P(A, b)$ is a basic solution corresponding to some basis.*

Proof: We know any vertex in standard form will have $k \leq m$ positive indices. Let $x_1, ..., x_k > 0$ and $x_{k+1}, ..., x_n = 0$. Further, the columns $1, ..., k$ of A must be linearly independent. Since we assume A is full rank, we can select $m - k$ additional linearly independent columns with index in $k + 1, ..., n$. Together all of these columns will give a basis for x. $\square$