1 Recap

- Finished proving strong duality
- Theorem of Alternatives

**Theorem 1 (Theorem of Alternatives)** Let \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \). Then exactly one of the following hold:

(a) \( \exists x \in \mathbb{R}^n \text{ such that } Ax \leq b \)
(b) \( \exists y \in \mathbb{R}^m \text{ such that } A^T y = 0, y \geq 0, b^T y < 0 \).

- Value function and sensitivity analysis:
  
  We showed
  
  \[ v(u) = \max\{c^T x | Ax \leq b + u\} \]

  is concave and piecewise linear.

2 Value Functions

**Proposition 2** Suppose that \( v(0) \) exists and is finite. Then the following are equivalent:

(a) \( \forall u \in \mathbb{R}^n, v(u) \leq v(0) + u^T y^* \);
(b) \( y^* \in \arg \min \{b^T y | A^T y = c, y \geq 0\} \).

**Proof:** Let \( x^*(u) \in Q(A, b + u) \) satisfy \( v(u) = c^T x^*(u) \).

“(a) \( \Rightarrow \) (b)”:

Let \( u^* \) satisfy (a). We will prove that

1. \( y^* \geq 0 \)
2. \( y^*(b - Ax^*(0)) = 0 \)
3. \( A^T y^* = c \)

Which implies that \( y^* \) is dual optimal because

\[
\min\{b^T y | A^T y = c, y \geq 0\} \geq c^T x^*(0) = (A^T y^*)^T x^*(0) = b^T y^* \geq \min\{b^T y | A^T y = c, y \geq 0\}.
\]
Proof of 1. \((y^* \geq 0)\)

\((\forall i) -\infty < v(0) \leq v(e_i)\) because \(Ax^*(0) \leq b \leq b + e_i\). Thus, the inequality \(v(e_i) \leq v(0) + e_i^T y^*\) implies

\[-e_i^T y^* \leq v(0) - v(e_i) \leq 0,\]

i.e., \(y_i \geq 0\).

Proof of 2. \((y^T(b - Ax^*(0)) = 0)\)

First, \(-\infty < v(0) \leq v(Ax^*(0) - b)\) because \(x^*(0) \in Q(A, b + (Ax^*(0) - b))\). Thus the inequality \(v(Ax^*(0) - b) \leq v(0) + (Ax^*(0) - b)^T y^*\) implies

\[0 \leq (b - Ax^*(0))^T y^* \leq v(0) - v(Ax^*(0) - b) \leq 0.\]

So \(y^T(b - Ax^*(0)) = 0\).

Proof of 3. \((A^T y^* = c)\)

Because \(x \in Q(A, b + (Ax - b))\), we have

\[(\forall x \in \mathbb{R}^n) \quad c^T x + (b - Ax)^T y^* \leq v(Ax - b) + (b - Ax)^T y^* \leq v(0),\]

where the second inequality follows by assumption. Notice that

\[v(0) = c^T x^*(0) = c^T x^*(0) + (b - Ax^*(0))^T y^*;\]

Thus \(f(x) = c^T x + (b - Ax)^T y^*\) is maximized at \(x^*(0)\). Therefore

\[0 = \nabla f(x^*(0)) = c - A^T y^*;\]

so \(A^T y^* = c\) and \(y^*\) is dual optimal.

“(a) \iff (b)”: If \(y^*\) is dual optimal then

\[v(0) = \max \{c^T x | Ax \leq b\} = b^T y^*.\]

If \(v(u) = -\infty\), then the inequality is trivial. Otherwise, because \(y^*\) is dual feasible it follows that

\[v(u) = \min \{(b + u)^T y | A^T y = c, \ y \geq 0\} \leq (b + u)^T y^* = v(0) + u^T y^*.\]

\[\square\]

Remarks

- We have shown that the set of optimal dual solutions is exactly \(-\partial[-V](0)\), i.e. the set of supgradients of \(v\) at 0. (See homework 3).
- By shifting \(v\), we have also shown that the subgradients of \(v\) at \(u\) are exactly the solutions of \(\min \{(b + u)^T y | A^T y = c, \ y \geq 0\}\).
- When \(v\) happens to be differentiable at \(u\), we can show that

\[\nabla v(u) = \arg \min \{(b + u)^T y | A^T y = c, \ y \geq 0\},\]

which implies that the dual minimizer is unique.
• Clearly, piecewise linear functions are differentiable almost everywhere on their domains (the almost everywhere part comes from ties in the expression for \( v \)). Notice that 

\[
\text{dom}(v) = \{u|v(u) > -\infty\} \supseteq \{u|u \geq 0\}.
\]

**Corollary 3**

\[
P(\arg \min \{(b + u)^T y | y \geq 0, A^T y = c\} \text{ is unique}) = 1
\]

where \( u \sim \text{Unif}[0,1]^m \).

## 3 Fourier Motzkin

Several times in this class we’ve made the assumption that, for linear maps \( L \),

\[
LQ(A,b) = \{x|Ax \leq b\}
\]

is a closed set. **This is a nontrival fact.** The next proof follows the proof from Jim Renegar’s Excellent textbook [1].

**Theorem 4 (Fourier Motzkin Elimination)** Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) and \( L \in \mathbb{R}^{k \times m} \). Then the set

\[
LQ(A,b) = \{x|Ax \leq b\}
\]

is a polyhedron.

**Proof:** Observation: It suffices to prove the result for a special kind of matrix, namely for \( P \) that project onto the first \( n - 1 \) components of \( x \).

\[
P = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 1 & 0
\end{bmatrix}
\]

**Why?** First by induction, if we show that \( PQ(A,b) \) is a polyhedron, then we can repeatedly remove any number of coordinates of \( x \) and still get a polyhedron. For example we could remove \( x_n, x_{n-1}, \ldots, x_{n-k} \) and the resulting set is still a polyhedron.

Then write down the set

\[
\hat{Q} = \left\{ \begin{bmatrix} y \\ x \end{bmatrix} \bigg| Ax \leq b, Lx - y \leq 0, -Lx + y \leq 0 \right\}.
\]

Notice that \( \begin{bmatrix} y & x \end{bmatrix}^T \in \hat{Q} \) if, and only if, \( Ax \leq b \) and \( Lx = y \).

If we remove the last \( n + m \) coordinates of all vectors in \( \hat{Q} \), the resulting set is still a polyhedron. Upon noticing that this set is \( LQ(A,b) = \left\{ Lx \bigg| Ax \leq b \right\} \) the conclusion of the theorem follows.
So let's work with $P$ which projects onto the first $n - 1$ coordinates. Let

\[
I_0 = \{ i \in \{1, \ldots, m\} \mid a_{in} = 0 \} \\
I_+ = \{ i \in \{1, \ldots, m\} \mid a_{in} > 0 \} \\
I_- = \{ i \in \{1, \ldots, m\} \mid a_{in} < 0 \}
\]

Then create a new matrix $\tilde{A} \in \mathbb{R}^{m \times n}$ and a new vector $\tilde{b} \in \mathbb{R}^m$.

\[
\tilde{A} = \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_m \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \end{bmatrix}
\]

where

\[
\tilde{a}_i = \begin{cases} a_i & \text{if } i \in I_0 \\ \frac{a_i}{a_{in}} & \text{if } i \in I_+ \cup I_- \end{cases}, \quad \tilde{b}_i = \begin{cases} b_i & \text{if } i \in I_0 \\ \frac{b_i}{a_{in}} & \text{if } i \in I_+ \cup I_- \end{cases}
\]

Then $\tilde{a}_i = [\gamma_i, \tau_i] \in \mathbb{R}^{1 \times m}$ with $\tau_i \in \{1, 0\}$. Let $\bar{x} = (x_1, \ldots, x_{n-1})$. Then $Q(A, b)$ is described by three types of inequalities.

\[
\gamma_i \bar{x} + 0x_n \leq \tilde{b}_i \quad i \in I_0 \\
\gamma_i \bar{x} + x_n \leq \tilde{b}_i \quad i \in I_+ \\
\gamma_i \bar{x} + x_n \geq \tilde{b}_i \quad i \in I_-.
\]

Then rearrange,

\[
0 \leq \tilde{b}_i - \gamma_i \bar{x} \quad i \in I_0 \\
x_n \leq \tilde{b}_i - \gamma_i \bar{x} \quad i \in I_+ \\
x_n \geq \tilde{b}_i - \gamma_i \bar{x} \quad i \in I_-.
\]

A point $\bar{x}$ is in $PQ(A, b)$ if, and only if, $\exists x_n \in \mathbb{R}$ s.t

\[
0 \leq \tilde{b}_i - \gamma_i \bar{x} \quad i \in I_0 \\
\max_{i \in I_-} \{\tilde{b}_i - \gamma_i \bar{x}\} \leq \min_{i \in I_+} \{\tilde{b}_i - \gamma_i \bar{x}\}
\]

i.e. if, and only if,

\[
0 \leq \tilde{b}_i - \gamma_i \bar{x}, \quad i \in I_0 \\
\tilde{b}_k - \gamma_k \bar{x} \leq x_n \leq \tilde{b}_i - \gamma_i \bar{x}, \quad \forall k \in I_-, \ i \in I_+.
\]

Thus $\bar{x} \in PQ(A, b)$ if and only if it satisfies a system of linear inequalities. Thus, $PQ(A, b)$ is a polyhedron.

\[\square\]

**Corollary 5** Consider two polyhedra $Q(A_1, b_1)$, $Q(A_2, b_2) \subseteq \mathbb{R}^n$, then $Q(A_1, b_1) + Q(A_2, b_2)$ and $Q(A_1, b_1) - Q(A_2, b_2)$ are polyhedra and therefore closed.
Remarks:

1. We implicitly used the fact that \( \{A^T | y \geq 0, y^T(b - Ax) = 0 \} \) was closed when proving the normal cone identities.

2. Calvin explicitly used the closedness of the above set to prove Farkas’ Lemma.

3. Fourier-Motzkin elimination is a specific example of something called ”Quantifier elimination.” Used to prove certain classes of sets are closed under a variety of operations.

References