1 Review

Last time we talked about normal cone of polyhedra:
\[ N_{\{x \mid Ax \leq b\}}(x) = \{ A^T y \mid y \geq 0, y^T (b - Ax) = 0 \} \]
\[ N_{\{y \mid y \geq 0, A^T y = c\}}(y) = \{-b \mid (\exists x) Ax \leq b, y^T (b - Ax) = 0 \} \]

We also proved strong duality in feasible case.

2 Strong Duality (Continued)

**Lemma 1** Let \( c \in \mathbb{R}^n \), suppose that \( p^* = \sup \{ c^T x \mid x \in Q(A,b) \} \) is finite, then \( \exists x^* \in Q(A,b) \), so that \( p^* = c^T x^* \)

**Proof:** Let \( S_1 = \{ x \mid c^T x = p^* \} \) and \( S_2 = Q(A,b), \). Suppose that \( S_1 \cap S_2 = \emptyset \). We have the following claim:

**Claim 2** \( S_1 - S_2 = \{ x - \bar{x} \mid c^T x = p^*, A\bar{x} \leq b \} \) is closed.

We will prove this claim in the next lecture.

Thus by Separating Hyperplane Theorem:
\[ \exists (\hat{a} \in \mathbb{R}^n \setminus \{0\}, \hat{b} \in \mathbb{R}), \text{ s.t. } \sup_{x \in S_1} \hat{a}^T x < \hat{b} < \inf_{y \in S_2} \hat{a}^T y. \]

We know \( \forall \varepsilon > 0, \exists x_\varepsilon \in S_2, \text{ s.t. } p^* - \varepsilon \leq c^T x_\varepsilon \leq p^*. \) Because \( S_1 \subset \{ x \mid \hat{a}^T x \leq \hat{b} \} \), we have \( \text{dist}(x_\varepsilon, S_1) \geq \text{dist}(x_\varepsilon, \{ x \mid \hat{a}^T x \leq \hat{b} \}) \) (where \( \text{dist}(x, S) = \| x - P_S(x) \| \) denotes the distance of a point to a closed convex set \( S \)).

We leave following two conclusions as exercises (\( x^+_\varepsilon := \max \{ x, 0 \} \))

1. \( P_{S_1}(x) = x - \frac{c^T x - p^*}{\| c \|^2} c; \)
2. \( P_{\{ x \mid \hat{a}^T x \leq \hat{b} \}}(x) = x - \frac{(\hat{a}^T x - \hat{b})_+}{\| \hat{a} \|^2} \hat{a}. \)

Thus
\[
\text{dist}(x_\varepsilon, S_1) = \| x_\varepsilon - P_{S_1}(x_\varepsilon) \|
= \| x_\varepsilon - \left( x_\varepsilon - \frac{c^T x - p^*}{\| c \|^2} c \right) \|
= \| \frac{c^T x_\varepsilon - p^*}{\| c \|} \| \leq \frac{\varepsilon}{\| c \|},
\]
and, by a similar argument,
\[
\text{dist}(x_\varepsilon, \{x|\hat{a}^T x \leq \hat{b}\}) = \|x_\varepsilon - P_{\{x|\hat{a}^T x \leq \hat{b}\}}(x_\varepsilon)\| = \frac{\hat{a}^T x_\varepsilon - \hat{b}}{\|\hat{a}\|}.
\]

By assumption, we have
\[
0 < \inf_{x \in S_2} \frac{\|\hat{a}^T x - \hat{b}\|}{\|\hat{a}\|} \leq \inf_{\varepsilon > 0} \frac{\|\hat{a}^T x_\varepsilon - \hat{b}\|}{\|\hat{a}\|} = \inf_{\varepsilon > 0} \text{dist}(x_\varepsilon, \{x|\hat{a}^T x \leq \hat{b}\}) \leq \inf_{\varepsilon > 0} \text{dist}(x_\varepsilon, S_1) \leq \inf_{\varepsilon > 0} \frac{\varepsilon}{\|c\|} = 0.
\]

We have reached a contradiction, so
\[
\exists x^* \in Q(A, b) \text{ s.t. } c^T x^* = p^*.
\]

\[\square\]

**Theorem 3** *(Strong Duality Infeasible Case)* Consider the linear programs:
\[
p^* = \max(c^T x | Ax \leq b), \quad d^* = \min(b^T y | A^T y = c, y \geq 0)
\]

3. \(d^* = \infty\) and primal is unbounded and \(p^* = \infty\),
4. \(p^* = -\infty\) and dual is unbounded and \(d^* = -\infty\)

**Proof of 3:**
(a) Suppose \(d^* = \infty\) and the primal is feasible. If \(\exists x^* \in Q(A, b), \text{ s.t. } c^T x^*\) is maximal, then \(c \in N_{Q(A,b)}(x^*) = \{A^T y|y \geq 0, y^T (b - Ax^*) = 0\}\). Any \(y\) that satisfies \(c = A^T y, y \geq 0\) is feasible for the dual. This contradicts the infeasibility of the dual. Thus \(p^* = \infty\) and \(Q(A, b)\) is unbounded.
(b) If \(Q(A, b)\) is unbounded, and \(p^* = \infty\), then by weak duality theorem, \(d^* \geq p^* = \infty\)

**Proof of 4:** We leave it as an exercise.

**3 Consequence of Strong Duality**

**Theorem 4** *(Theorem of Alternatives)* Exactly one of the following hold:

1. \(\exists x, \text{ s.t. } Ax \leq b\),
2. \(\exists y, \text{ s.t. } A^T y = 0, b^T y < 0, y \geq 0\).

**Proof:** We firstly prove 1 and 2 cannot hold simultaneously. If \(x\) satisfies 1 and \(y\) satisfies 2, then \(0 = y^T (Ax) \leq y^T b < 0\).
Secondly, suppose 1 is false, then \(\max\{s|Ax + s1 \leq b, s \leq 0\}\) always has a solution (we are maximizing a negative number). Write this program in matrix form:
\[
\begin{bmatrix}
A & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
s
\end{bmatrix}
\leq
\begin{bmatrix}
b \\
0
\end{bmatrix}
\]
This linear program has an optimal solution \((x^*, s^*)\) and optimal value \(s^* < 0\). By strong duality, the dual
\[
\min b^T y \\
\text{s.t.} \begin{bmatrix} A^T & 0 \\ 1^T & 1 \end{bmatrix} \begin{bmatrix} y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
\(y \geq 0, t \geq 0\)
has an optimal solution \((y^*, t^*)\) with optimal value \(b^T y^* = s^* < 0\) and \(A^T y^* = 0, y^* \geq 0\).

Now suppose 2 is false, then 1 cannot be false (otherwise 2 would be true), hence 1 is true. \(\square\)

4  Sensitivity Analysis and Value Function

**Definition 1** (Maximal Value Function) \(v(u) = \max\{c^T x | Ax \leq b + u\}\).

We have following two natural questions:

1. Can we bound the value function in terms of \(v(0)\)? If \(v(u)\) is particularly expensive to compute, knowing a bound on it in terms of \(v(0)\) can help us determine whether it might be worth it to re-solve the linear program.

2. What is the rate of change, i.e., the derivative of \(v\)?

**Lemma 5** Suppose \(v(0)\) exists and \(x^*(0) \in Q(A, b)\) satisfies \(c^T x^*(0) = p^*\). Let \(\text{dom}(v) := \{u | v(u) > -\infty\}\). Then following three hold:

1. \(\forall u \in \mathbb{R}^m, v(u) < \infty\);
2. \(v\) is concave;
3. \(v\) is piecewise linear.

**Proof of 1:** Since \(x^*(0)\) exists, we know \(\exists y_0 \geq 0\), s.t. \(A^T y_0 = c\) (by strong duality), so \(V(u) = \max \{c^T x | Ax \leq b + u\} \leq \min\{ (b + u)^T y_0 | y_0 \geq 0, A^T y_0 = c \} \leq (b + u)^T y_0 < \infty\) (by weak duality). \(\square\)

**Proof of 2:** Let \(u_1, u_2 \in \text{dom}(v)\) and let \(\lambda \in [0, 1]\). By strong duality,
\[
v(\lambda u_1 + (1 - \lambda) u_2) = \min\{ (b + \lambda u_1 + (1 - \lambda) u_2)^T y | y \geq 0, A^T y = c \}
\]
\[
= \min\{ \lambda (b + u_1)^T y + (1 - \lambda) (b + u_2)^T y | y \geq 0, A^T y = c \}
\]
\[
\geq \lambda \min\{ (b + u_1)^T y | y \geq 0, A^T y = c \} + (1 - \lambda) \min\{ (b + u_2)^T y | y \geq 0, A^T y = c \}
\]
\[
= \lambda v(u_1) + (1 - \lambda) v(u_2).
\]
\(\square\)

**Proof of 3:** By the results of Recitation 4, we have, for all \(u \in \text{dom}(v)\),
\[
v(u) = \min\{ (b + u)^T y | A^T y = c, y \geq 0 \}
\]
\[
= \min\{ (b + u)^T y_k | y_1, \ldots, y_E \text{ are extreme points of } \{y | A^T y = c, y \geq 0\} \}.
\]
\(\square\)