1 Review

A while back, we defined polyhedrons and polytopes as follows.

**Definition 1** A Polyhedron is $P = \{x \in \mathbb{R}^n : Ax \leq b\}$

**Definition 2** A Polytope is given by $Q = \text{conv}(v_1, v_2, ..., v_k)$, where the $v_i$ are the vertices of the polytope, for $k$ finite.

Also recall the equivalence of extreme points, vertices and basic feasible solutions, and recall the definition of a bounded polyhedron.

**Definition 3** A polyhedron $P$ is bounded iff $\exists M > 0$ such that $\|x\| \leq M$, $\forall x \in P$.

We showed bounded polyhedra were polytopes by taking the extreme points and seeing that they were the vertices for $P$ as a polytope.

Recall also the Separating Hyperplane Theorem from a previous lecture.

**Theorem 1** (Separating Hyperplane) Let $C \subseteq \mathbb{R}^n$ be a closed, nonempty and convex set. Let $y \in \mathbb{R}^n \setminus C$ and let

$$x^* = P_C(y) := \arg\min_{x \in C} \frac{1}{2}\|x - y\|^2.$$

Then there exists a number $b \in \mathbb{R}$, such that with $a = y - x^*$, we have

$$(\forall x \in C) \quad a^T x \leq a^T x^* < b < a^T y.$$  

2 The polar of a set

Now we want to prove that polytopes are bounded polyhedra. To do this, we need to introduce one more concept.

**Definition 4** Let $S \subseteq \mathbb{R}^n$. We call the set

$$S^\circ = \{z \in \mathbb{R}^n : z^T x \leq 1, \forall x \in S\},$$

the polar set of $S$.  

Example 1 (Polars of the $\ell_\infty$ and $\ell_1$ balls.) Consider $S = \{(x_1, x_2) : -1 \leq x_1 \leq 1, \ -1 \leq x_2 \leq 1\}$, the region is shown in the left of Figure 1.

By definition, $x \in S^\circ$ if, and only if, $|x_1| + |x_2| = \sup_{z_1, z_2 \in S} \{x_1 z_1 + x_2 z_2\} \leq 1$. Thus, $S^\circ = \{(x_1, x_2) : -1 \leq x_1 \leq 1, \ -1 \leq x_2 \leq 1\}$, which is shown on the right hand side of Figure 1.

Now let’s consider $S^{\circ\circ}$. By definition, $x \in (S^\circ)^\circ$ if, and only if, $\max\{|z_1|, |z_2|\} = \sup_{|x_1|+|x_2|\leq1} \{x_1 z_1 + x_2 z_2\} \leq 1$. Thus, $(S^\circ)^\circ = \{(x_1, x_2) : -1 \leq x_1 \leq 1, \ -1 \leq x_2 \leq 1\} = S$.

Lemma 2 If $C$ is a closed convex subset of $\mathbb{R}^n$ with $0 \in C$, then $C^{\circ\circ} := (C^\circ)^\circ = C$.

Proof:

• ($\supseteq$) Suppose that $x \in C$. Then $x \in C^{\circ\circ}$ if, and only if, $z^T x \leq 1$ for all $z \in C^\circ$. This is clearly true because $z \in C^\circ$ implies that $z^T x \leq 1$.

• ($\subseteq$) We will show that if $x \notin C$, then $x \notin C^{\circ\circ}$. First note that $C$ is closed and convex with at least $z = 0 \in C$. If $x \notin C$, then by the Separating Hyperplane Theorem, there exists $0 \neq a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ with $a^T x > b > a^T z$ for all $z \in C$. Since $0 \in C$, we have $b > 0$. Let $\tilde{a} = a/b \neq 0$. Therefore $\tilde{a}^T x > 1 > \tilde{a}^T z$, for all $z \in C$. This implies $\tilde{a} \in C^\circ$. But $\tilde{a}^T x > 1$, so $x \notin C^{\circ\circ}$.

Therefore $C^{\circ\circ} = C$. □

3 Polytopes are Bounded Polyhedra

Now we can prove our result, at least sort of. We’ll assume that 0 is in the interior of the polytope. We claim that this can be done without loss of generality; this is because we can translate the polytope to have 0 $\in P$, apply the following proof and then translate back if needed.

Theorem 3 If $Q \subseteq \mathbb{R}^n$ is a polytope with 0 in the interior of $Q$, then $Q$ is a (bounded) polyhedron.

Proof: Our proof strategy is as follows. (1) We will first show that the polar of a polytope is a polyhedron. (2) We then show that since the polytope has 0 in its interior, then the polar of the polytope is bounded. So then $P = Q^\circ$ is a bounded polyhedron. (3) We know from a previous lecture that any bounded polyhedron is a polytope, so $P = Q^\circ$ is a polytope. (4) But then applying the proof that the polar of a polytope is a polyhedron, we get that $P^\circ = Q^{\circ\circ} = Q$ (by the lemma above) is a polyhedron. It is easy to prove that a polytope is bounded. Thus, we must prove (1) and (2).
(1) We first prove that the polar of $Q$ is a polyhedron. Let $P = Q^\circ$. Then we know that $P^\circ = Q^{\circ\circ} = Q$. Since $Q$ is a polytope, $Q = \text{conv}\{v_1, \ldots, v_k\}$ for some $k$ finite vectors $v_1, \ldots, v_k \in \mathbb{R}^n$. We claim that
\[ P = \{ z \in \mathbb{R}^n : v_i^T z \leq 1, \ i = 1, \ldots, k \}. \]

One the one hand, $P = Q^\circ = \{ z \in \mathbb{R}^n : x^T z \leq 1, \ \forall x \in Q \}$, so $v_i^T z = z^T v_i \leq 1$ for $i = 1, 2, \ldots, k$, which implies that $P \subseteq \{ z \in \mathbb{R}^n : v_i^T z \leq 1, \ i = 1, \ldots, k \}$. On the other hand, if $z^T v_i \leq 1$ for $i = 1, \ldots, k$, then for any $x \in Q$, with $x = \sum_{i=1}^{k} \lambda_i v_i$ for some $\lambda_i \geq 0$, $\sum \lambda_i = 1$, we have
\[ z^T x = z^T \sum_{i=1}^{k} \lambda_i v_i = \sum_{i=1}^{k} \lambda_i (z^T v_i) \leq \sum_{i=1}^{k} \lambda_i = 1, \]
which proves the claim. Thus $P$ is a polyhedron.

(2) Because $0 \in \text{int}(Q)$, there exists some $\epsilon > 0$ such that all $x \in \mathbb{R}^n$ with $\|x\| < \epsilon$ lie in $Q$. If $z \in P$, $z \neq 0$, then, because $\|x\| < \epsilon$, we have
\[ x = \frac{\epsilon}{2 \|z\|} \frac{z}{\|z\|} \in Q. \]

Because $P = Q^\circ$,
\[ x^T z \leq 1 \Rightarrow \frac{\epsilon z^T z}{2 \|z\|} \leq 1 \Rightarrow \|z\| \leq \frac{2}{\epsilon}. \]
Hence $P$ is a bounded polyhedron. \qed

4 Normal Cone

Modern optimization theory crucially relies on a concept called the normal cone.

**Definition 5** Let $S \subset \mathbb{R}^n$ be a closed, convex set. The normal cone of $S$ is the set-valued mapping $N_S : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, given by
\[ N_S(x) = \begin{cases} \{ g \in \mathbb{R}^n | (\forall z \in S) \ g^T (z - x) \leq 0 \} & \text{if } x \in S \\ \emptyset & \text{if } x \notin S \end{cases} \]

![Figure 2: Normal cones of several convex sets.](image)
Example 2 We compute several normal cones; see Figure 2.

1. Let $S = \{ z \}$.
   \[
   N_S(x) = \begin{cases} 
   \mathbb{R}^n & \text{if } x = z \\
   \emptyset & \text{otherwise}
   \end{cases}
   \]

2. Let $S = [0, 1]$.
   \[
   N_S(x) = \begin{cases} 
   \mathbb{R}_{\leq 0} & \text{if } x = 0 \\
   \mathbb{R}_{\geq 0} & \text{if } x = 1 \\
   \{0\} & \text{if } x \in (0, 1) \\
   \emptyset & \text{otherwise}
   \end{cases}
   \]

3. Let $S = \{ x \mid \|x\| \leq 1, \ x \in \mathbb{R}^n \}$.
   \[
   N_S(x) = \begin{cases} 
   \mathbb{R}_{\geq 0} & \text{if } \|x\| = 1 \\
   \{0\} & \text{if } \|x\| < 1 \\
   \emptyset & \text{otherwise}
   \end{cases}
   \]

4. The normal cone of a triangle, computed at some but not all points, is depicted in Figure 2.

Normal cones satisfy several useful properties.

Proposition 4 Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed, convex set. Then the following hold:

1. If $x \in S$, then $N_S(x)$ is a convex cone, i.e.
   \[
   (\forall \lambda_1 \geq 0), (\forall \lambda_2 \geq 0), (\forall g_1 \in N_S(x)), (\forall g_2 \in N_S(x)) \lambda_1 g_1 + \lambda_2 g_2 \in N_S(x).
   \]

2. Let $y \in \mathbb{R}^n \setminus S$, then $P_S(y) = x \iff y - x \in N_S(x)$.

3. If $x \in \text{int}(S)$, then $N_S(x) = \{0\}$.

4. If $x \in S$ and $x \notin \text{int}(S)$, then $N_S(x) \supseteq \{0\}$.

Proof:

1. We leave the proof of part 1 as an exercise.

2. ($\Rightarrow$): By separating hyperplane theorem, with $a = y - P_S(y) = y - x$, $\exists b \in \mathbb{R}$, s.t.
   \[
   (\forall z \in S) \quad a^T z \leq a^T P_S(y) \leq a^T y
   \]
   \[
   \Rightarrow \quad a^T (z - P_S(y)) \leq 0
   \]
   \[
   \Rightarrow \quad a \in N_S(P_S(y))
   \]

($\Leftarrow$): If $y - x \in N_S(x)$, then

\[
\forall z \in S \quad (y - x)^T(z - x) \leq 0
\]

\[
\iff \quad (y - x)^T(y - x) + (y - x)^T(z - y) \leq 0
\]

\[
\Rightarrow \quad \|y - x\|^2 \leq (y - x)^T(y - z) \leq \|y - x\| \|y - z\| \quad (\text{Cauchy - Schwarz inequality})
\]

\[
\Rightarrow \quad \|y - x\| \leq \|y - z\|, \quad \forall z \in S
\]

\[
\Rightarrow \quad x = P_S(y)
\]

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3. Suppose that $g \in N_S(x)$. Because $x \in \text{int}(S)$, there exists $\epsilon > 0$, such that $x + \epsilon g \in S$. Therefore, we have

$$g^T((x + \epsilon g) - x) \leq 0$$
$$\Rightarrow \epsilon g^T g \leq 0$$
$$\Rightarrow g = 0.$$

Hence, $N_S(x) = \{0\}$.

4. Since $x \notin \text{int}(S)$, there exists a sequence $y^k \in \mathbb{R}^n \setminus S$, s.t. $y^k \rightarrow x$ as $k \rightarrow \infty$. We leave it as an exercise to prove that if $x^k = P_S(y^k)$, then $x^k \rightarrow x$, $k \rightarrow \infty$.

Let $g^k = \frac{y^k - x^k}{\|y^k - x^k\|}$. Then, by part 5, we have $g^k \in N_S(x^k)$. Without loss of generality, we can assume that $g^k \rightarrow g \in \mathbb{R}^n$, with $\|g\| = 1$.

We claim that $g \in N_S(x)$. To prove the claim, note that since $g^k \in N_S(x^k)$, we have

$$(g^k)^T(z - x^k) \leq 0,$$

and

$$g^T(z - x) = (g - g^k)^T(z - x) + (g^k)^T(z - x)$$
$$= (g - g^k)^T(z - x) + (g^k)^T(x^k - x) + (g^k)^T(z - x^k)$$
$$\leq (g - g^k)^T(z - x) + (g^k)^T(x^k - x)$$
$$\rightarrow 0 \text{ as } k \rightarrow \infty$$

So for all $z \in S$, we have $g^T(z - x) \leq 0$, which means $g \in N_S(x)$. Obviously $0 \in N_S(x)$, so $\{0\} \subsetneq N_S(x)$.

Proposition 5 shows that normal cones detect the boundary and interior of convex sets.

**Corollary 5** Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed, convex set. Then

- $N_S(x) = \{0\}$ if, and only if, $x \in \text{int}(S)$.
- $N_S(x) \not\supseteq \{0\}$ if, and only if, $x \in S \setminus \text{int}(S)$.