Lecture 3

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Last time we discussed how to take dual of an LP in two different ways. Today we will talk about the geometry of linear programs.

1 Geometry of Linear Programs

First we need some definitions.

Definition 1 *A set $S \subseteq \Re^n$ is* convex *if $\forall x, y \in S$, $\lambda x + (1 - \lambda)y \in S$, $\forall \lambda \in [0, 1]$.*
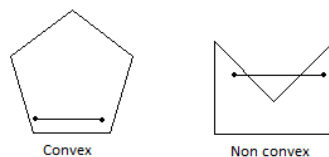

Figure 1: Examples of convex and non convex sets

Given a set of inequalities we define the feasible region as $P = \{x \in \Re^n : Ax \le b\}$. We say that P is a *polyhedron*.

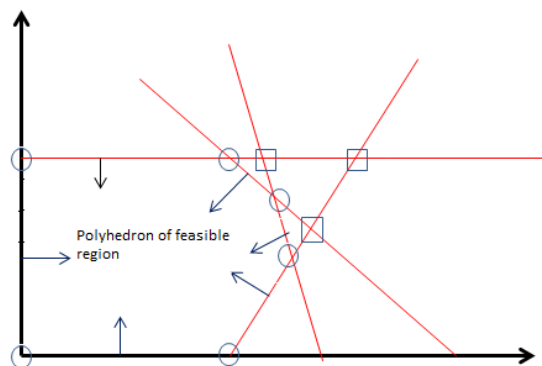Which points on this figure can have the optimal value? Our intuition from last time is that

Figure 2: Example of a polyhedron. "Circled" corners are feasible and "squared" are non feasible

optimal solutions to linear programming problems occur at "corners" of the feasible region. What we'd like to do now is to consider formal definitions of the "corners" of the feasible region.

One idea is that a point in the polyhedron is a corner if there is some objective function that is minimized there uniquely.

Definition 2 $x \in P$ is a vertex of P if $\exists c \in \Re^n$ with $c^T x < c^T y, \forall y \neq x, y \in P$.

Another idea is that a point $x \in P$ is a corner if there are no small perturbations of x that are in P.

Definition 3 Let S be a convex set in $\Re^n$. Then $x \in S$ is an extreme point of S if x cannot be written as $\lambda y + (1 - \lambda)z$ for $y, z \in S$, $y, z \neq x$, $0 \leq \lambda \leq 1$.

Note that it is possible that these two definitions are equivalent under some condition. Observe that for a unit disk, the points on the boundary are both extreme points and vertices. Given any x on the boundary of the disk, the vector $c = -x$ satisfies $c^T x < c^T y$ for all $\|y\| \leq 1$ with $y \neq x$. However, because these definitions are generalized for all convex sets - not just polyhedra - a point could possibly be extreme but not be a vertex. One set of examples are the points on an oval where the line segments of the sides meet the curves of the ends.
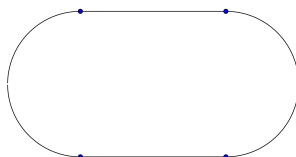

Figure 3: *Four extreme points in a two-dimensional convex set that are not vertices.*

A final possible definition is an algebraic one. We note that a corner of a polyhedron is characterized by a point at which several constraints are simultaneously satisfied. For any given x, let $A_=$ be the constraints satisfied with equality by x; (that is, a_i such that $a_i x = b_i$). Let $A_<$ be the constraints a_i such that $a_i x < b_i$.

Definition 4 Call $x \in \Re^n$ a basic solution of P if $A_=$ has rank n. x is a basic feasible solution of P if it also lies inside P (so each constraint is either in $A_=$ or $A_<$).

Since there are only a finite number of constraints defining P, there are only a finite number of ways to choose $A_=$, and if $rank(A_=) = n$ then x is uniquely determined by $A_=$. So there are at most $\binom{m}{n}$ basic solutions. Now we want to show that all these definitions are equivalent.

Theorem 1 (*Characterization of Vertices*). *Let P be defined as above. The following are equivalent:*

(1) x is a vertex of P.

(2) x is an extreme point of P.

(3) x is a basic feasible solution of P.

Proof: We first prove that $(1) \Rightarrow (2)$. Let x be a vertex of P and suppose by way of contradiction that x is not an extreme point of P. Since x is a vertex, $\exists c \in \Re^m$ such that $c^T x < c^T y$ for all $y \in P, y \neq x$. Because x is not an extreme point, there exist $y, z \in P$, $y, z \neq x$, $0 \leq \lambda \leq 1$ such that $x = \lambda y + (1 - \lambda)z$. Therefore $c^T x < c^T y$ and $c^T x < c^T z$. Thus

$$c^T x < \lambda c^T y + (1 - \lambda)c^T z = c^T(\lambda y + (1 - \lambda)z) = c^T x.$$

This gives us a contradiction, so x must be an extreme point.

We now prove $(2) \Rightarrow (3)$, by proving the contrapositive, that $\neg(3) \Rightarrow \neg(2)$. If x is not a basic feasible solution and $x \in P$, then the column rank of $A_=$ is less than n. Hence there is a direction vector $0 \neq y \in \Re^n$ such that $A_= y = 0$ (i.e. the columns of $A_=$ are linearly dependent). We want to show that for some $\varepsilon > 0$, $x + \varepsilon y \in P$ and $x - \varepsilon y \in P$. Then we will have shown that x can be written as a convex combination of two other points of P, since then $x = \frac{1}{2}(x + \varepsilon y) + \frac{1}{2}(x - \varepsilon y)$, which contradicts x being an extreme point. To show that $x + \varepsilon y \in P$ and $x - \varepsilon y \in P$, we want to show that

(a) $A_=(x + \varepsilon y) \leq b_=$.

(b) $A_<(x + \varepsilon y) \leq b_<$.

(c) $A_=(x - \varepsilon y) \leq b_=$.

(d) $A_<(x - \varepsilon y) \leq b_<$.

For the first inequality we have that

$$A_=(x + \varepsilon y) = A_= x + \epsilon A_= y = A_= x = b_=$$

since $A_= y = 0$. Showing the third is similar.

For proving second we can find the appropriate $\varepsilon > 0$, we first note that since $A_< x < b_<$, $b_< - A_< x > 0$, so we can choose a small $\varepsilon > 0$ such that $\varepsilon A_< y \leq b_< - A_< x$ and $-\varepsilon A_< y \leq b_< - A_< x$. Thus for showing the second inequality we note that

$$A_<(x + \varepsilon y) = A_< x + \epsilon A_< y \leq A_< x + (b_< - A_< x) = b_<$$

by our choice of ε. Showing the fourth inequality is similar.

Finally, we prove $(3) \Rightarrow (1)$. Let $I = \{i : a_i x = b_i\}$. Set $c = -\sum_{i \in I} a_i^T$. Then

$$c^T x = \sum_{i \in I} a_i x = -\sum_{i \in I} b_i,$$

and for any $y \in P$,

$$c^T y = -\sum_{i \in i} a_i y \geq -\sum_{i \in I} b_i = c^T x$$

by the feasibility of y. Then it must be the case that $c^T y = c^T x$ only if $a_i y = b_i$ for all $i \in I$. Thus $A_= y = b_=$. However, since x is a basic feasible solution, $A_=$ has rank n, so that $A_= x = b_=$ has a unique solution x. Then if $c^T x = c^T y$, it must be the case that $x = y$. Hence we have that $c^T x = c^T y$ implies that $x = y$ and $c^T x \leq c^T y$ for all $y \in P$, and thus x is a vertex. $\square$

2   Convex Hulls

We now look at another way of specifying a feasible region.

Definition 5 *Given $v_1, v_2, \ldots, v_k \in \Re^n$, a convex combination of $v_1, v_2, \ldots, v_k$ is $v = \sum_{i=1}^{k} \lambda_i v_i$ for some λ_i such that $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.*

Definition 6 *For $Q = \{v \in \Re^n : v$ is a convex combination of $v_1, v_2, \ldots, v_k\}$, we say that Q is a convex hull of $v_1, v_2, \ldots, v_k$, and we write $Q = conv(v_1, v_2, \ldots, v_k)$.*

Definition 7 *For Q the convex hull of a finite number of vectors $v_1, v_2, \ldots, v_k$, Q is a* <u>*polytope*</u>.

Now we will do some exercises to develop intuition about convex hulls.

Lemma 2 *Q is convex.*

Proof: Pick any $x, y \in Q$. This implies that

$$x = \sum_{i=1}^{k} \alpha_i v_i \quad \alpha_i \geq 0 \quad \sum_{i=1}^{k} \alpha_i = 1,$$
$$y = \sum_{i=1}^{k} \beta_i v_i \quad \beta i \geq 0 \quad \sum_{i=1}^{k} \beta i = 1.$$

For $\lambda \in [0, 1]$, then

$$
\begin{aligned}
\lambda x + (1 - \lambda) y &= \lambda \sum_{i=1}^{k} \alpha_i v_i + (1 - \lambda) \sum_{i=1}^{k} \beta_i v_i \\
&= \sum_{i=1}^{k} [\lambda \alpha_i + (1 - \lambda) \beta_i] v_i.
\end{aligned}
$$

Then we know that $\lambda \alpha_i + (1 - \lambda) \beta_i \geq 0$ for all i, and that

$$\sum_{i=1}^{k} (\lambda \alpha_i + (1 - \lambda) \beta_i) = \lambda \sum_{i=1}^{k} \alpha_i + (1 - \lambda) \sum_{i=1}^{k} \beta_i = 1.$$

Thus

$$\lambda x + (1 - \lambda) y = \sum_{i=1}^{k} \delta_i v_i,$$

where $\delta_i = \lambda \alpha_i + (1 - \lambda) \beta_i$, so that $\delta_i \geq 0$ for all i, and $\sum_{i=1}^{k} \delta_i = 1$. Thus $\lambda x + (1 - \lambda) y \in Q$.   □