Dantzig-Wolfe Decompositions [1]

Dantzig-Wolfe decomposition (1960) is an algorithm for solving large-scale optimization problems, which have a special structure (block structure).

We consider an LP as follows.

$$\begin{array}{llllllll} \min & c_1^T x_1 & + & c_2^T x_2 & + & \dots & + & c_m^T x_m & \\ \text{s.t.} & A_{01}x_1 & + & A_{02}x_2 & + & \dots & + & A_{0m}x_m & = b_0 \\ & A_{11}x_1 & & & & & & & = b_1 \\ & & & A_{22}x_2 & & & & & = b_2 \\ & & & & & \dots & & & \\ & & & & & & & A_{mm}x_m & = b_m \\ & & & & & & & x_1, \dots, x_m & \geq 0 \end{array}$$

where $c_j \in \mathbb{R}^{n_j}$, $b_0 \in \mathbb{R}^{m_0}$ and $b_i \in \mathbb{R}^{m_i}$. The constraint $A_{01}x_1 + A_{02}x_2 + \dots + A_{0m}x_m = b_0$ is called a linking constraint. We can consider an example of this LP, such as a company which has many divisions. Each division has its own decision variables and constraints, and the company also sets constraints on their shared resources. This LP can be rewritten more concisely as follows.

$$\begin{aligned} \min \quad & \sum_{i=1}^m c_i^T x_i \\ \text{s.t.} \quad & \sum_{i=1}^m A_{0i} x_i = b_0 && (*) \\ & A_{ii} x_i = b_i \qquad i = 1, \dots, m \\ & x_i \geq 0 \qquad i = 1, \dots, m \end{aligned}$$

Exploit that it is relatively easy to solve the following problem for any vector $\hat{c}$.

$$\begin{array}{lll} \min & \hat{c}^T x_i & \\ \text{s.t.} & A_{ii} x_i & = b_i \\ & x_i & \geq 0 \end{array}$$

Let $Q_i = \{x_i \in \mathbb{R}^{n_i} : A_{ii}x_i = b_i, x_i \geq 0\}$. Assuming that Q_i is bounded, we could in theory enumerate its vertices $\{v_{i1}, v_{i2}, \dots, v_{iN_i}\}$. (If Q_i is not bounded, we would also need extreme rays.) Thus, any $x_i \in Q_i$ can be written as $x_i = \sum_{j=1}^{N_i} \lambda_{ij} v_{ij}$ for some λ_{ij} such that $\sum_{j=1}^{N_i} \lambda_{ij} = 1$ and $\lambda_{ij} \geq 0$.

[1] Based on previous notes of Jeff Tian

Threrefore, $(*)$ can be rewritten as

$$\begin{array}{llll} \min & \sum_{i=1}^{m}\sum_{j=1}^{N_i}\lambda_{ij}(c_i^T v_{ij}) & & \\ \text{s.t.} & \sum_{i=1}^{m}\sum_{j=1}^{N_i}\lambda_{ij}(A_{0i}v_{ij}) & = b_0 & \\ & \sum_{j=1}^{N_i}\lambda_{ij} & = 1 & i = 1, ..., m \\ & \lambda_{ij} & \geq 0 & \forall i, j. \end{array}$$

We call this the master problem. Notice that there are now $m_0 + m$ constraints but many more variables. Now, we want to use column generation to solve the master problem. Assume we have some basic feasible solution (BFS) to the master problem. We can compute the dual variables $\begin{bmatrix} y \\ z \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^{m_0} \\ \mathbb{R}^{m} \end{bmatrix}$ using complementary slackness.

The reduced cost of variable λ_{ij}:

$$\begin{aligned} \overline{c_i^T v_{ij}} &= c_i^T v_{ij} - \begin{bmatrix} A_{0i}v_{ij} \\ e_i \end{bmatrix}^T \begin{bmatrix} y \\ z \end{bmatrix} \\ &= c_i^T v_{ij} - (A_{0i}v_{ij})^T y - e_i^T z \\ &= c_i^T v_{ij} - (v_{ij}^T A_{0i}^T) y - z_i \\ &= (c_i - A_{0i}^T y)^T v_{ij} - z_i. \end{aligned}$$

Notice that $\overline{c_i^T v_{ij}} < 0$ is equivalent to $(c_i - A_{0i}^T y)^T v_{ij} < z_i$. Let $\overline{c_i} = c_i - A_{0i}^T y$. Now consider the following problem.

$$\begin{array}{lll} \min & \bar{c}_i^T x_i & \\ \text{s.t.} & A_{ii}x_i & = b_i \\ & x_i & \geq 0. \end{array}$$

Since Q_i is nonempty and bounded, this problem has an optimal solution that is a vertex, say v_{ik} for some k. If $\bar{c}_i^T v_{ik} < z_i$, then v_{ik} should enter the basis for the master problem, else $\overline{c_i^T v_{ij}} \geq 0$ for all vertices v_{ij}. So by solving the subproblem for each $Q_1, ..., Q_m$, we either find a negative reduced cost variable, or prove that the current solution is optimal.

We reduce the space required to solve the overall problem, and each subproblem could possibly be solved in parallel. What we have done is just simplex method plus column generation for finding negative reduced costs.