In class, we saw that every bounded polyhedra is a polytope in the set of convex combination of its vertices. Now we will extend the theory to pointed polyhedra (i.e., those that contain no lines).

**Definition 1** A polyhedron \( Q \) is pointed if there is no \( y \in Q \) and \( d \neq 0 \) such that for any \( \lambda \in \mathbb{R} \), \( y + \lambda d \in Q \).

**Definition 2** Let \( C \) be a nonempty convex set: then the recession cone of \( C \), \( \text{rec}(C) \), is
\[
\{ d \in \mathbb{R}^m : \forall x \in C, \forall \alpha \geq 0, x + \alpha d \in C \}.
\]

**Proposition 1** If \( C \) is a nonempty set then \( \text{rec}(C) \) is a nonempty convex cone.

**Proof:** Let \( d_1, d_2 \in \text{rec}(C), \lambda_1, \lambda_2 \geq 0 \). We want to show that \( \lambda_1 d_1 + \lambda_2 d_2 \in \text{rec}(C) \). For any \( x \in C \) and any \( \alpha \geq 0 \)
\[
x + \alpha(\lambda_1 d_1 + \lambda_2 d_2) = [x + (\alpha \lambda_1) d_1] + (\alpha \lambda_2) d_2.
\]
The quantity in brackets lies in \( C \) since \( \alpha \lambda_1 \geq 0 \) and \( d_1 \in \text{rec}(C) \), and then the desired vector lies in \( C \) because \( \alpha \lambda_1 \geq 0 \) and \( d_2 \in \text{rec}(C) \). Also, \( 0 \in \text{rec}(C) \) by definition. \( \square \)

**Proposition 2** For \( Q := \{ y \in \mathbb{R}^m : Ax \leq b \} \) then (if \( Q \) is nonempty)
\[
\text{rec}(Q) = \{ d \in \mathbb{R}^m : Ad \leq 0 \}.
\]

**Proof:**
\( \supseteq \):
if \( Ad \leq 0 \) then for any \( y \in Q, \alpha \geq 0 \).
\[
A(x + \alpha d) = Ax + \alpha Ad \\
\leq b + 0 \\
= b,
\]
hence \((y + \alpha d) \in Q \).

\( \subseteq \):
Suppose \( d \in \text{rec}(Q) \), and choose any \( y \in Q \). Then \( \forall \alpha \geq 0 \)
\[
A(x + \alpha d) = Ax + \alpha Ad \\
\leq c \\
\]
This implies \( Ax \leq b \) and \( Ad \leq 0 \), otherwise, the above would fail for large \( \alpha \). \( \square \)

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1 Based on previous notes of Chaoxu Tong
Theorem 3 (Representation of Pointed Polyhedra). Let $Q$ (defined as in Proposition 2) be a nonempty pointed polyhedron, and let $P$ be the set of all convex combinations of its vertices and $K$ be its recession cone. Then

$$Q = P + K := \{p + d : p \in P, d \in K\}.$$

Proof:

$\supseteq$:

Any $p \in P$ is inside $Q$ and, thus, satisfies all linear constraints of $Q$, so any $p + d \in P + K$ has

$$A^T(p + d) = A^T_p + A^T d \leq c + 0 = c.$$

$\subseteq$:

The proof is by induction on $\{m - ra(y)\}$.

True for $\{m - ra(y) = 0\} \iff y$ is itself a vertex of $Q$ and $d = 0 \in \text{rec}(C)$.

Suppose true if $\{m - ra(y) < k\}$ for some $k > 0$ and consider $y \in Q$ with $ra(y) = m - k < m$. Choose $0 \neq d \in \mathbb{R}^m$ with $A^T_y d = 0$ ($A^T_y$ are all equality constraints for $y$) and consider $y + \alpha d, \alpha \in \mathbb{R}$.

Since $Q$ is pointed there are three cases to consider.

(1) $\alpha$ is bounded above and below, say by $\underline{\alpha} < 0 \ & \ \overline{\alpha} > 0$.

As in the previous theorem

$$y = \frac{\overline{\alpha}}{\overline{\alpha} - \alpha}(y + \alpha d) + \frac{-\alpha}{\overline{\alpha} - \alpha}(y + \overline{\alpha}d),$$

and $(y + \overline{\alpha}d)$ has $m - ra(y + \overline{\alpha}d) < k$, so

$$(y + \overline{\alpha}d) = p + \overline{d}, \ p \in P, \ \overline{d} \in K,$$

and similarly

$$(y + \alpha d) = p + d, \ p \in P, \ d \in K,$$

so

$$y = \frac{\overline{\alpha}}{\overline{\alpha} - \alpha}(p + d) + \frac{-\alpha}{\overline{\alpha} - \alpha}(\overline{p} + \overline{d}) + \ldots d + \ldots d.$$

The vector in brackets is a point of $P$ and that in braces a point in $K$.

(2) $\alpha$ is bounded below but not above. Then $d \in K$ and $y = [y + \alpha d] + (-\alpha)d$, with $\alpha$ defined as before. The vector in brackets lies in $P + K$ as in the first part by the inductive hypothesis. Therefore

$$y = (p + d) + (-\alpha)d$$

$$= \overline{p} + (d + (-\alpha)d)$$

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lies in $P + K$.

(3) $\alpha$ is bounded above but not below. Then we can simply switch $d$ to $-d$ and $\alpha$ to $-\alpha$, and we get back to case (2).

This completes the proof. \qed

**Theorem 4** (Fundamental theorem of LP). Consider the LP problem $\max \{ b^T y : y \in Q \}$ with $Q$ being a pointed polyhedron. Then

1. if there is a feasible solution, there is a vertex solution (basic feasible solution);
2. if there is a feasible solution and $b^T y$ is unbounded above on $Q$, then there is a ray or halfline: \{ $y + \alpha d : \alpha \geq 0$ \} $\in$ $Q$ on which $b^T y$ is unbounded above; and
3. if $b^T y$ is bounded above on $Q$, then the max is attained and attained at a vertex $Q$.

**Proof:**
(1): If $Q \neq \emptyset$, $P \neq \emptyset$, so there exists a vertex.

(2) & (3):
 Assume $P \neq \emptyset$ & $P$ is a set of convex combinations of $v_1, v_2, v_3, ..., v_k$.

$$\sup \{ b^T y : y \in Q \} = \sup \{ b^T y : y \in P + K \} = \sup \{ b^T p + b^T d : p \in P, d \in K \} = \sup \{ b^T p : p \in P \} + \sup \{ b^T d : d \in K \}.$$ 

If there is some $\vec{d} \in K$ with $b^T \vec{d} > 0$ then by considering $\alpha \vec{d}$, $\alpha \to +\infty$, see that $\sup \{ b^T d : d \in K \} = +\infty$. Then $b^T y$ is unbounded above on $Q$ and clearly unbounded above on \{ $y + \alpha \vec{d} : \alpha \geq 0$ \} for any $y \in Q$.

If there is no such $\vec{d} \in K$, then $\sup \{ b^T d : d \in K \} = 0$, attained by $d = 0$. Then

$$\sup \{ b^T y : y \in Q \} = \sup \{ b^T p : p \in P \} = \sup \{ \sum_{i=1}^{k} \lambda_i (b^T v_i) : \sum_{i=1}^{k} \lambda_i = 1, \ all \ \lambda_i \geq 0 \} = \max_{1 \leq i \leq k} b^T v_i.$$ 

In this case $\sup \{ b^T y : y \in Q \}$ is attained by $y = v_i$ where $i$ attains the maximum. \qed