ORIE 6300 Mathematical Programming I	September 7, 2016

Recitation 3

Lecturer: Calvin Wylie — *Topic: Farkas' Lemma*

Feasibility and Unboundedness [1]

Consider a linear program in arbitrary form. We know that it can potentially be infeasible or have unbounded optimal objective. Additionally, if it's feasible and not unbounded, we can show the existence of an optimal solution by applying the Weierstrass Theorem. Hence, this gives three options for the types of solutions a linear program can have.

Additionally, the dual of a linear program is itself a linear program, so the same three options apply. Hence, the first natural question is what combinations of these can appear for a primal-dual pair of linear programs?

$P\backslash D$	I	O	U
I	?	?	?
O	?	?	?
U	?	?	?

Let's try to fill in some of these boxes.

First, recall the weak duality theorem: If x is a feasible solution to a minimization linear program and y is a feasible solution to its dual, then $b^T y \leq cx$.

Suppose the primal minimization program is unbounded. This immediately implies that the dual must be infeasible. Similarly, if the dual is unbounded, this immediately implies that the primal must be infeasible. To see this in the first case, let y be any feasible solution to the dual. Since the primal is unbounded, there exists an $\hat{x}$ such that $c\hat{x} < b^T y$, contradicting the Weak Duality Theorem. Hence, no such y exists. The other argument can be proved identically.

Hence, our table now looks like:

$P\backslash D$	I	O	U
I	?	?	✓
O	?	?	X
U	✓	X	X

[1] Based on previous notes of Chaoxu Tong

Given the above theorem, it seems natural to ask whether the reverse implication holds. Does primal infeasibility imply dual unboundedness? Consider the following LP:

$$\begin{array}{ll} \max & [2, -1]\ x \\ \text{s.t.} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ & x \geq 0 \end{array}$$

Its corresponding dual is:

$$\begin{array}{ll} \max & [-1, 2]\ y \\ \text{s.t.} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} y \leq \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ & y \geq 0 \end{array}$$

Note that the primal is infeasible and that the dual feasible region is exactly the primal feasible region, hence, both are infeasible. This adds another option to our table, giving:

Finally, using Strong Duality Theorem we know when one of primal or the dual has an optimal solution, they both must have an optimal solution. Hence our table looks like:

$P\backslash D$	I	O	U
I	✓	X	✓
O	X	✓	X
U	✓	X	X

Farkas Lemma and its Application

First recall the Farkas' Lemma :

Theorem 1 (Farkas' Lemma) *If $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, then exactly one of the following holds:*

1. $\exists x \geq 0$ *such that* $Ax = b$
2. $\exists y$ *such that* $A^T y \geq 0, b^T y < 0$

Proof: First we show that we can't have both (1) and (2). Assume for contradiction $\exists \hat{x}$ such that $A\hat{x} = b, \hat{x} \geq 0$, and $\exists \hat{y}$ such that $A^T\hat{y} \geq 0$, $y^T b \leq 0$. Note that $\hat{y}^T A\hat{x} = \hat{y}^T(A\hat{x}) = \hat{y}^T b < 0$ since by (1), $A\hat{x} = b$ and by (2) $\hat{y}^T b < 0$. But also $\hat{y}^T A\hat{x} = (\hat{y}^T A)\hat{x} = (A^T\hat{y})^T\hat{x} \geq 0$ since by (2) $A^T\hat{y} \geq 0$ and by (1) $\hat{x} \geq 0$.

Now we must show that if (1) doesn't hold, then (2) does. To do this, let $v_1, v_2, \ldots, v_n$ be the columns of A. Define

$$Q = cone(v_1, \ldots, v_n) \equiv \{s \in \Re^m : s = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \forall i\}.$$

This is a conic combination of the columns of A, which differs from a convex combination since we don't require that $\sum_{i=1}^{n} \lambda_i = 1$. Then $Ax = \sum_{i=1}^{n} x_i v_i$, there exists an x such that $Ax = b$ and $x \geq 0$ if and only if $b \in Q$ as x's are weights λ_i.

So if (1) does not hold then $b \notin Q$. We show that condition (2) must hold. We know that Q is nonempty (since $0 \in Q$), closed (see Lecture 8), and convex, so we can apply the separating hyperplane theorem. The theorem implies that there exists $\alpha \in \Re^m$, $\alpha \neq 0$, and β such that $\alpha^T b > \beta$ and $\alpha^T s < \beta$ for all $s \in Q$. Since $0 \in Q$, we know that $\beta > 0$. Note also that $\lambda v_i \in Q$ for all $\lambda > 0$. Then since $\alpha^T s < \beta$ for all $s \in Q$, we have $\alpha^T(\lambda v_i) \in Q$ for all $\lambda > 0$, so that $\alpha^T v_i < \beta/\lambda$ for all $\lambda > 0$. Since $\beta > 0$, as $\lambda \to \infty$, we have that $\alpha^T v_i \leq 0$. Thus by setting $y = -\alpha$, we obtain $y^T b < 0$ and $y^T v_i \geq 0$ for all i. Since the v_i are the columns of A, we get that $A^T y \geq 0$. Thus condition (2) holds. □

Here is another form of the Farkas Lemma:

Theorem 2 (Alternative Farkas' Lemma) *Exactly one of the following holds:*

1. $\exists x \geq 0$ *such that* $Ax \leq b$
2. $\exists y$ *such that* $A^T y \geq 0, b^T y < 0, y \geq 0$

Proof: Note $Ax \leq b$, $x \geq 0$ is feasible if and only if $Ax + s = b$, $x, s \geq 0$ is feasible. Apply the original Farkas Lemma to this new system. □

Let's see an application of the Farkas Lemma. Note that we can only prove that unboundedness implies infeasibility for linear programs and not the converse in the previous section. We now prove a related implication for the unboundedness of feasible **regions** of linear programs.

Theorem 3 (Clark's Theorem) *Given the following primal and dual LPs, if one of them is feasible, then the feasible region for one of them is non-empty and unbounded.*

Primal LP:		*Dual LP:*	
min	$c^T x$	*max*	$y^T b$
s.t.:	$Ax \geq b$	*s.t.:*	$A^T y \leq c$
	$x \geq 0$		$y \geq 0$

It's important to note that the result of the theorem is that the feasible region of one of the LPs is unbounded, but it may not be the case that the LP has unbounded objective function value with the given objective function.

Proof: There are three possibilities to consider.

1. The primal is infeasible and the dual is unbounded.
2. The dual is infeasible and the primal is unbounded.
3. Both the primal and dual are feasible and not unbounded (hence have optimal solution).

In the first two cases, we immediately have the result we want. Hence, suppose we're in the last case. Now, let $\hat{c} = [-1, -1, \ldots, -1]$ and consider the following systems:

1. $\exists \hat{y}$ such that $A^T \hat{y} \leq \hat{c}, \hat{y} \geq 0$

2. $\exists \hat{x}$ such that $A\hat{x} \geq 0, \hat{c}^T\hat{x} < 0$, $\hat{x} \geq 0$

Alternative Farkas' lemma tells us exact one of them holds.

If (2) holds, let $\hat{x}$ be a feasible solution to (2) and x be a feasible solution to primal LP and $\lambda > 0$. Note $\hat{x}$ is not 0 because $\hat{c}^T\hat{x} < 0$. Then

$$A(x + \lambda\hat{x}) = Ax + \lambda A\hat{x} \geq b + \lambda * 0 = b$$

So $x + \lambda\hat{x}$ is feasible for all $\lambda \geq 0$. Feasible region for primal LP is unbounded and we're done.

Otherwise if (1) holds, let $\hat{y}$ be a feasible solution to (1) (note $\hat{y}$ is not 0 because $A^T\hat{y} \leq \hat{c}$) and let y be a feasible solution to dual LP. Then using a similar argument as above, we can show that for any $\lambda \geq 0$, $y + \lambda\hat{y}$ is feasible for the dual, which shows the feasible region for dual LP is unbounded and we're done. $\square$