ORIE 6340: Mathematics of Data Science  
Homework 1  
(Last Updated February 11, 2019)  

Damek Davis*

1. Let $K$ be a finite subset of the Euclidean ball in $\mathbb{R}^n$. Show that 
\[ w(K) \lesssim \sqrt{\log(|K|)}. \]

2. Let $K$ consist of all unit $s$-sparse vectors in $\mathbb{R}^n$:  
\[ K := \{ x \in \mathbb{R}^d \mid \|x\|_2 = 1, \|x\|_0 \leq s \}. \]

Show that 
\[ w(K) \lesssim \sqrt{s \log(2n/s)}. \]

3. (Minkowski averages are nearly convex) Suppose that $S \subseteq \mathbb{R}^n$ and $\text{diam}(S) \leq 1$. For any $k \in \mathbb{N}$, consider the set 
\[ S_k = \frac{1}{k}(S + S + \cdots S) = \left\{ \frac{1}{k} \sum_{i=1}^{k} x_i \mid x_1, \ldots, x_k \in S \right\}. \]

Show that 
\[ \text{dist}_H(\text{conv}(S), S_k) \leq \frac{1}{\sqrt{k}}, \]

where $\text{dist}_H$ denotes the Hausdorff distance between sets in $\mathbb{R}^n$.

4. Suppose $K$ is a polytope on $N$ vertices and $\text{diam}(K) \leq 1$. Show that $K$ can be covered by at most $N^{[1/2]}$ balls of radii $\varepsilon$.

5. Show that the Gaussian width is invariant under translations, orthogonal transformations, and taking convex hulls.

6. By direct calculation, show that the stable dimension of a bounded set is always bounded by the algebraic dimension (see lecture notes for definition).

---

*School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14850, USA; people.orie.cornell.edu/dsd95/.
7. Let $K = \{\pm 1\}^n$. Given measurement vectors $a_i \in \mathbb{N}(0, I_n)$ ($i = 1, \ldots, m$), suppose that we wish to recover a vector $x \in K$ (up to error) from measurements

$$y_i = \langle a_i, x \rangle \quad i = 1, \ldots, m. \quad (0.1)$$

Use the $M^*$ bound to compute an upper bound on the expected error of any estimator $\hat{x}$ in $K$ satisfying $A\hat{x} = y$, where $A$ is matrix whose $i$th row is $a_i^T$.

Can you provide a lower bound on

$$\mathbb{E} \left[ \sup_{x \in K} \|x - \hat{x}\|_2 \right]?$$