A $\frac{3}{2}$-Approximation Algorithm for Some Minimum-Cost Graph Problems

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Joint work with James M. Davis, Cornell University

23 August 2012
ISMP 2012
Berlin, Germany

A Simple Graph Problem

MIN WCF(k)

Input:

- Undirected graph $G = (V, E)$;
- Edge costs $c(e) \geq 0$ for all $e \in E$;
- Positive integer k.

Goal: Find a minimum-cost forest F such that each component has at least k vertices.

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$k = 2 \Rightarrow$ minimum edge-cover problem.
$k = n \Rightarrow$ minimum spanning tree problem.
$k \geq 3$, constant $\Rightarrow$ NP-hard (Imielińska, Khachiyan, Kalantari 1993, Bazgan, Couëtoux, Tuza, 2011)

Approximation Algorithms

Definition

An α-approximation algorithm is a polynomial-time algorithm that returns a solution of cost at most α times the cost of an optimal solution.

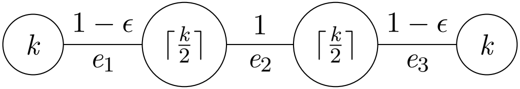
A 2-Approximation Algorithm

A component is *small* if $< k$ vertices, *big* otherwise.

$F \leftarrow \emptyset$
while F is not a feasible solution **do**
    Let e be the cheapest edge joining two comps C_1, C_2, at least one small
    $F \leftarrow F \cup \{e\}$
Return F

Due to Imielińska, Khachiyan, Kalantari 1993.

A Tight Example



Circles cliques of cost zero edges. Algorithm returns $\{e_1, e_3\}$ of cost $2-\epsilon$, optimal is $\{e_2\}$ of cost 1.

A Generalization

Goemans and W 1994 generalize to functions $h : 2^V \to \{0,1\}$. Want min-cost edges F such that $|\delta(S) \cap F| \geq h(S)$ for all $S \subset V$, where $\delta(S)$ is set of edges with exactly one endpoint in S.

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Other Problems

Gives a 2-approximation algorithm if h is *downwards monotone*: $h(T) = 1 \Rightarrow h(S) = 1$ for all $S \subseteq T$.

MIN WCF(k): $h(S) = 1$ if $|S| < k$.

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MIN WCF(k): $h(S) = 1$ if $|S| < k$.

Another example: Depots $D \subseteq V$, cost $c(d)$. Find min-cost edges F, depots D' such that each component has at least k vertices, at least 1 open depot.

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In 2011, Couëtoux gives a $\frac{3}{2}$-approximation algorithm for MIN WCF(k), simple modification of Imielińska et al. algorithm.

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Main Idea:

- Each edge added reduces number of small components.
- Should be willing to pay twice as much for edge that eliminates *two* small components.
- Edge e *good* if it connects two small components, results in large component; *bad* edge if it connects two components, at least one small.

Couëtoux's Algorithm

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  Let e' be the cheapest bad edge; joins two comps, at
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  if good e exists and $c(e) \leq 2c(e')$ **then**
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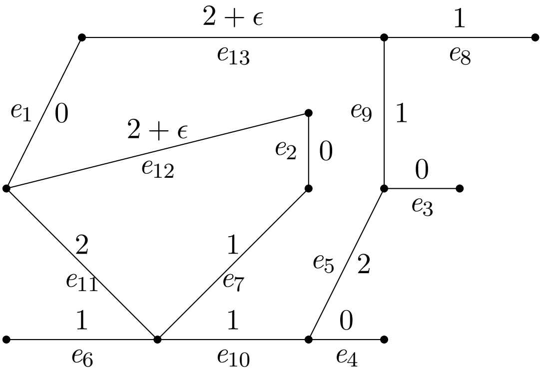
Example for $k = 4$



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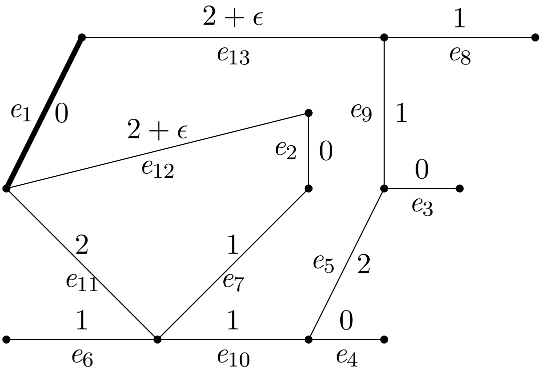

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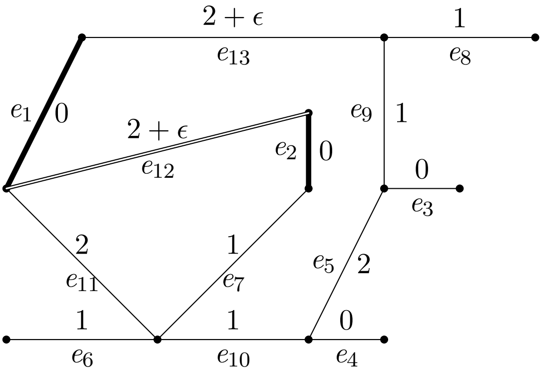

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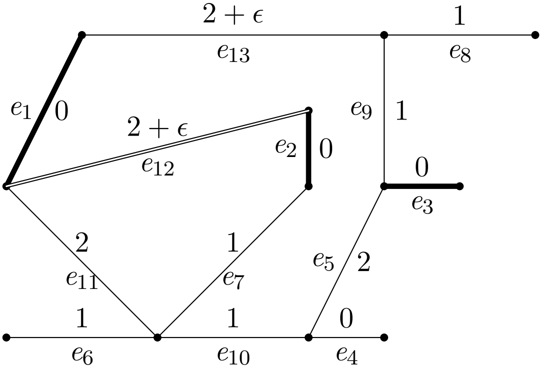

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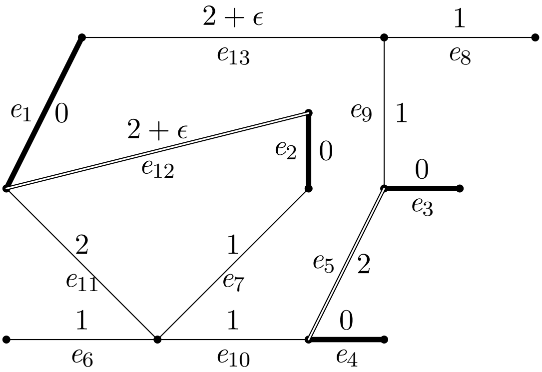

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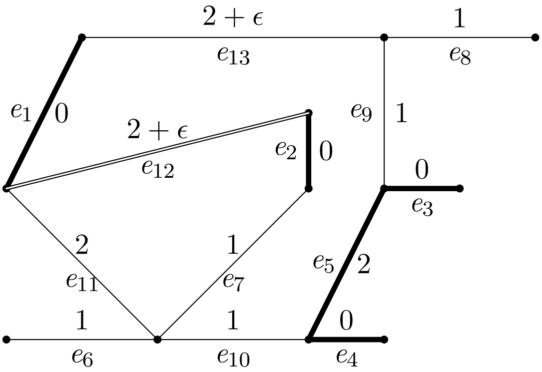

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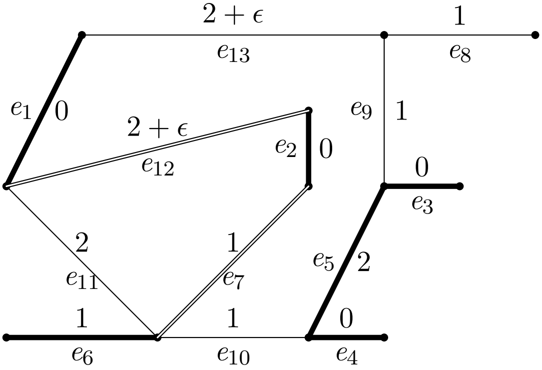

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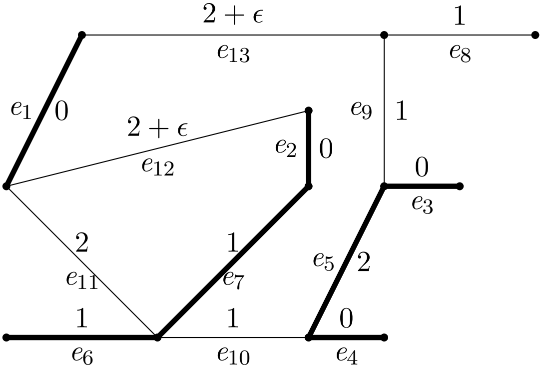

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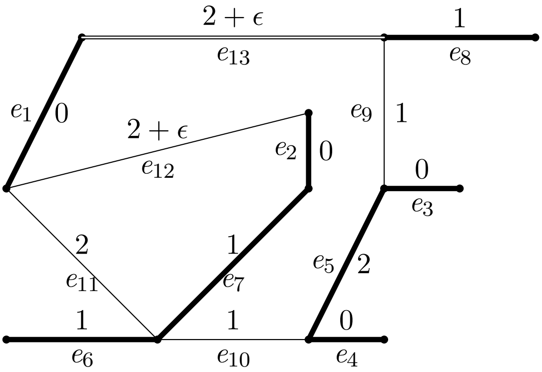

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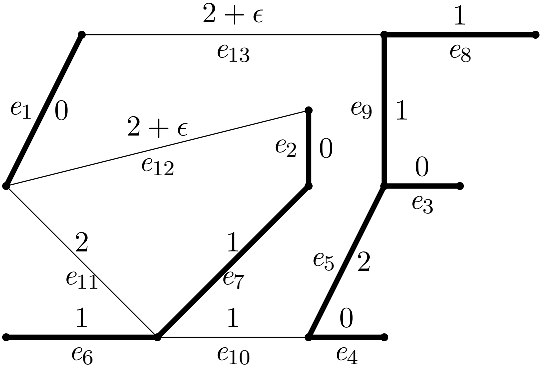

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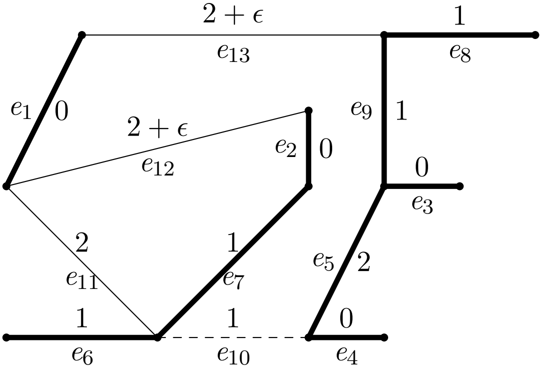

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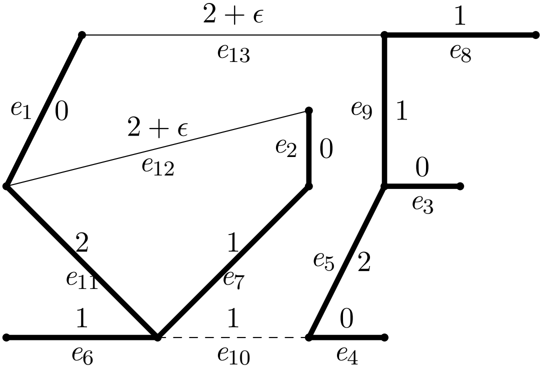

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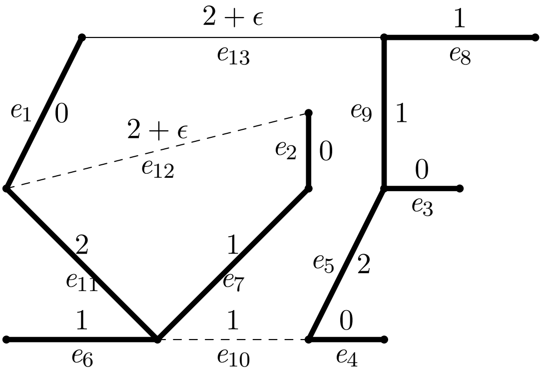

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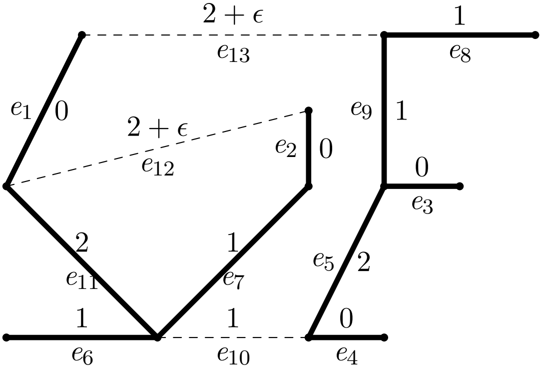

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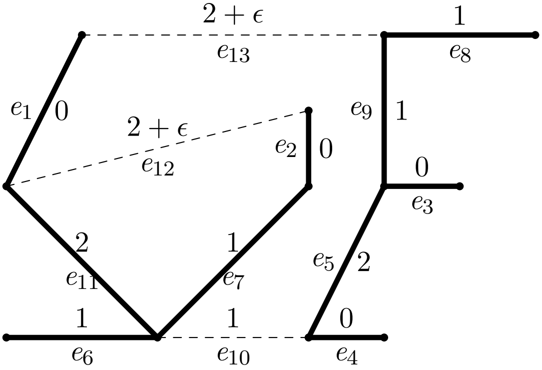
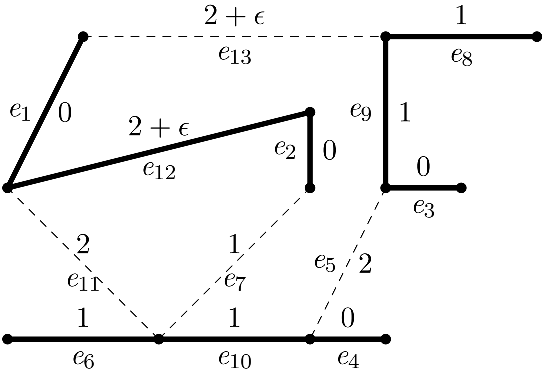


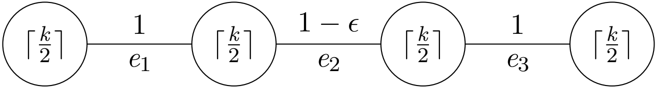
Cost is 8.

Optimal Solution



Optimal cost is $6 + \epsilon$.

A Tight Example



Circles cliques of cost zero edges. Algorithm returns $\{e_2, e_1, e_3\}$ of cost $3 - \epsilon$, optimal is $\{e_1, e_3\}$ of cost 2.

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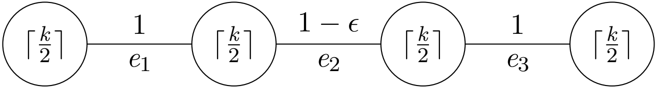

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Analysis: Complicated charging scheme.

Our Contributions

- We extend Couëtoux's algorithm to downwards monotone functions (easy: small $C \Rightarrow h(C) = 1$, large $C \Rightarrow h(C) = 0$)
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Main idea: Generate a "dual" solution of value at least cost of algorithm's solution. Show that $2/3$ of dual solution is a lower bound on any feasible solution.

Our algorithm

$F \leftarrow \emptyset$
while F is not a feasible solution **do**
  Let e be the cheapest good edge (if such an edge exists);
    joins two comps C_1, C_2 with $h(C_1) = h(C_2) = 1$,
    $h(C_1 \cup C_2) = 0$
  Let e' be the cheapest bad edge; joins two comps C_1, C_2,
    $\max(h(C_1), h(C_2)) = 1$
  if good e exists and $c(e) \leq 2c(e')$ **then**
    $F \leftarrow F \cup \{e\}$
  else
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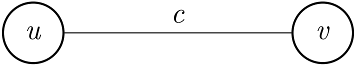
Return F

Ideas of the analysis

For simplicity, assume there is no vertex v such that $h(\{v\}) = 0$.

First, make a mixed graph by adding arcs for every edge.



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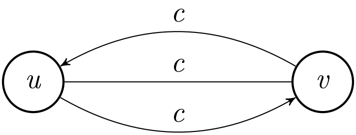
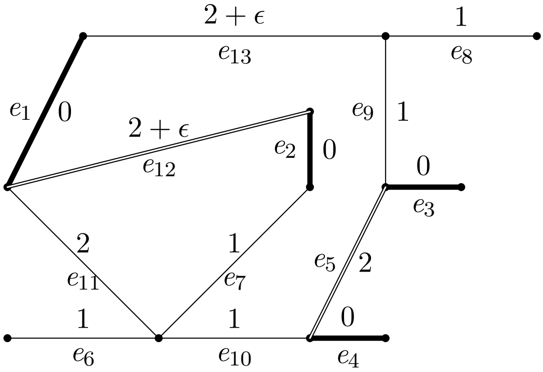


Birooted Components

Note: every component in the algorithm's solution has exactly one good edge; the edge added when the connected component first had some large subcomponent.

For analysis, treat good edge as undirected, all other edges in component directed towards the two endpoints of the good edge. Call the component *birooted*.

Back to the Example



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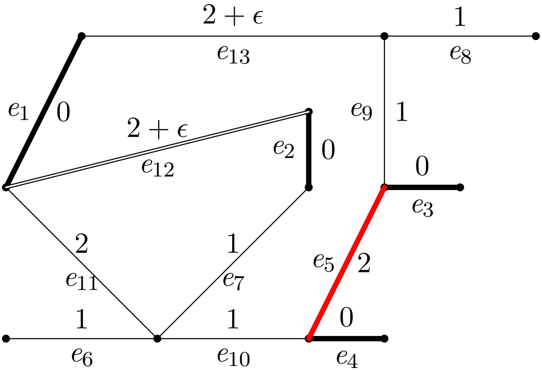

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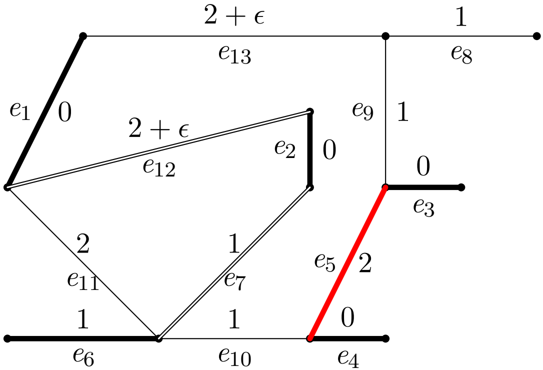

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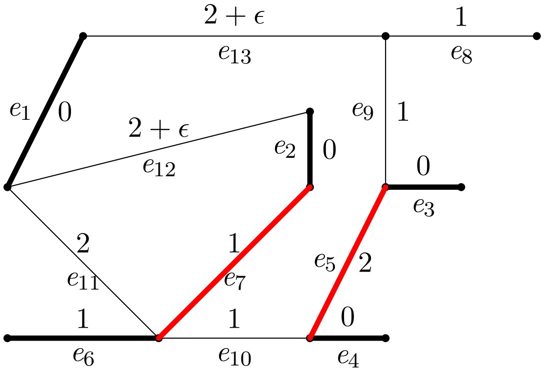

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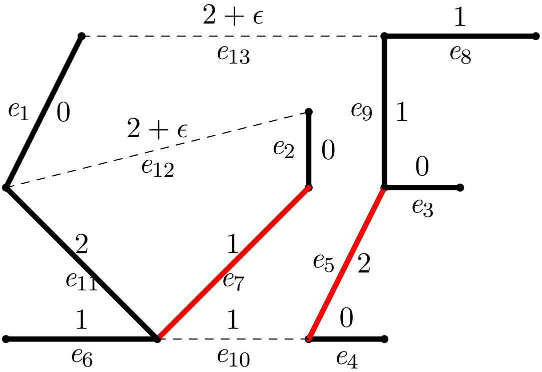

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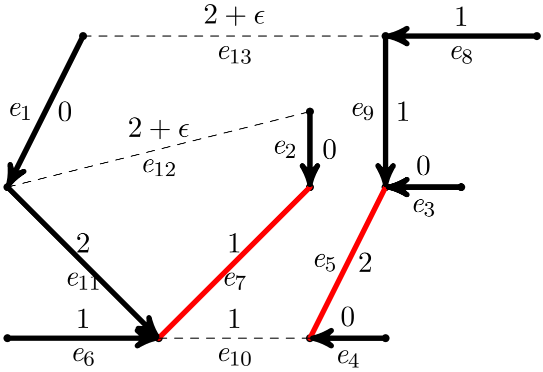

"Duals"

Introduce variables $y(S)$ for each $S \subseteq V$; will have

$$\sum_{S: a \in \delta^+(S)} y(S) \leq c(a)$$

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Inequality may be violated for *edges* e; may have

$$\sum_{S: e \in \delta(S)} y(S) > c(e).$$

The Algorithm, Again

$F \leftarrow \emptyset$
$y \leftarrow 0$
while F is not a feasible solution **do**
  Increase $y(C)$ for all components C with $h(C) = 1$ until either:
  (1) $\sum_{S: e \in \delta(S)} y(S) \geq c(e)$ for some good edge e; OR
  (2) $\sum_{S: a' \in \delta^+(S)} y(S) = c(a')$ for some bad arc $a' \equiv$ bad edge e';
  if (1) happens **then**
    $F \leftarrow F \cup \{e\}$
  else
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Return F

Back to Analysis

Lemma

For birooted component C constructed by the algorithm, can show that cost of C at most

$$\sum_{S \subseteq C} y(S).$$

For each arc a in C, $\sum_{S: a \in \delta^+(S)} y(S) = c(a)$, and for good edge e,

$$\sum_{e \in \delta(S)} y(S) \geq c(e).$$

Key Lemma

Say that a component $C \in \delta(S)$ if some edge of C is in $\delta(S)$.

Lemma

For any feasible solution F^*, and any component C^* of F^*,

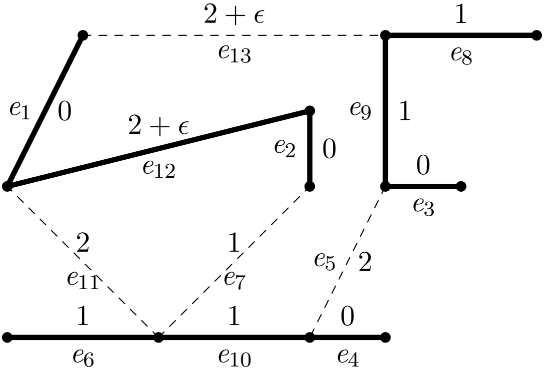
$$\sum_{S: C^* \in \delta(S)} y(S) \leq \frac{3}{2} \sum_{e \in C^*} c(e).$$

Then let F^* be an optimal solution, $\mathcal{C}^*$ its components, F the algorithm's solution. Then

$$\begin{aligned}\sum_{e \in F} c(e) \leq \sum_{S} y(S) &\leq \sum_{C^* \in \mathcal{C}^*} \sum_{S: C^* \in \delta(S)} y(S) \\ &\leq \sum_{C^* \in \mathcal{C}^*} \left(\frac{3}{2} \sum_{e \in C^*} c(e) \right) \\ &= \frac{3}{2} \sum_{e \in F^*} c(e).\end{aligned}$$

Proof Ideas for Key Lemma

For each component C^* of solution F^*, identify a good edge and biroot the component.



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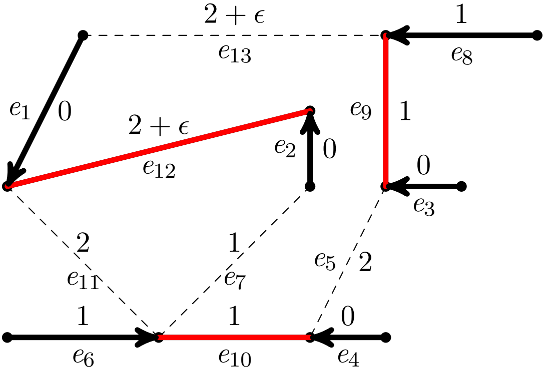

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To prove:

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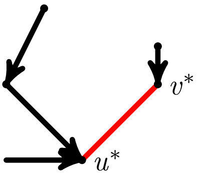
Let $e^* = (u^*, v^*)$ be good edge of birooted component.

Since $\sum_{S: a \in \delta^+(S)} y(S) \leq c(a)$ for each arc a in birooted component, only need to bound $\sum_{S: u^* \text{ or } v^*} y(S)$.



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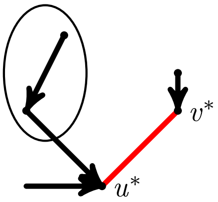

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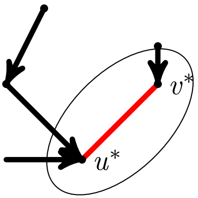

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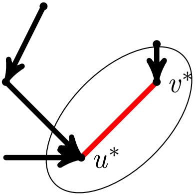

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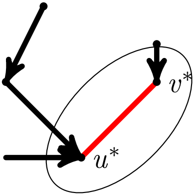

If $\sum_{S:u^*\in S} y(S) + \sum_{S:v^*\in S} y(S) \leq \frac{3}{2} c(e^*)$, then done.

Otherwise, argue that C^* must have an edge $e' \neq e^*$ such that $\sum_{S:u^*\in S} y(S) + \sum_{S:v^*\in S} y(S) \leq \frac{1}{2} c(e') + \frac{3}{2} c(e^*)$. Then also done.

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Thank you for your attention.