ORIE 6334 Approximation Algorithms February 27, 2014

Problem Set 3

Due Date: March 13, 2014

1. W&S Exercise 5.4

2. W&S Exercise 6.2

3. In this exercise, we'll show how a randomized greedy algorithm can also be used to give a $\frac{3}{4}$-approximation algorithm for the maximum satisfiability problem. Assume the variables of the instance are $x_1, \ldots, x_n$; in iteration i, we'll set the ith variable to true or false. Let SAT_i denote the total weight of the clauses satisfied by the settings of the variables at the end of iteration i, and let UNSAT_i be the total weight of the clauses that cannot be satisfied given the settings of the variables at the end of iteration i (that is, the clauses only contain literals from $x_1, \ldots, x_i$ and given the settings of the variables the clauses are not satisfied). Let W be the total weight of all clauses. Then observe that SAT_i is a lower bound on the weight of clauses satisfied by the algorithm, while $W - \text{UNSAT}_i$ is an upper bound.

   Let $\text{SAT}_{i,t}$ be SAT_i with x_i set true, $\text{SAT}_{i,f}$ be SAT_i with x_i set false, and similarly for $\text{UNSAT}_{i,t}$ and $\text{UNSAT}_{i,f}$. Let $B_i = \frac{1}{2}(\text{SAT}_i + (W - \text{UNSAT}_i))$, $B_{i,t} = \frac{1}{2}(\text{SAT}_{i,t} + (W - \text{UNSAT}_{i,t}))$, and $B_{i,f} = \frac{1}{2}(\text{SAT}_{i,f} + (W - \text{UNSAT}_{i,f}))$. The bound B_i is midway between our lower and upper bounds on the total weight that the algorithm will satisfy. We'd like to set the next variable so as to increase this bound. In iteration i, we'll compute $t_i = B_{i,t} - B_{i-1}$ and $f_i = B_{i,f} - B_{i-1}$. If $t_i \geq 0$ and $f_i < 0$, we set x_i true. If $t_i \leq 0$ and $f_i \geq 0$, we'll set x_i false. Otherwise (if $t_i > 0$ and $f_i > 0$) we'll set x_i true with probability $t_i/(t_i + f_i)$. Note the similarity with the "double-sided greedy" algorithm for nonmonotone submodular function maximization that we saw earlier.

   (a) Show that for any i, $t_i + f_i \geq 0$.

   (b) Given an optimal solution x^*, let OPT_i be the assignment in which $x_1, \ldots, x_i$ are set according to the algorithm and $x_{i+1}, \ldots, x_n$ are set as in x^*. Let $w(\text{OPT}_i)$ be the weight of the clauses satisfied by OPT_i. Then show that for any i,

   $$E[w(\text{OPT}_{i-1}) - w(\text{OPT}_i)] \leq \max\left(0, \frac{2t_i f_i}{t_i + f_i}\right).$$

   (c) Show that for any i,

   $$E[w(\text{OPT}_{i-1}) - w(\text{OPT}_i)] \leq E[B_i - B_{i-1}].$$

(d) Show that the algorithm is a $\frac{3}{4}$-approximation algorithm for MAX SAT.