1 The Matrix-Tree Theorem

In this lecture, we continue to see the usefulness of the graph Laplacian, and its connection to yet another standard concept in graph theory, that of a spanning tree. Let $A[i]$ be the matrix $A$ with its $i^{th}$ column and row removed. We will give two different proofs of the following.

**Theorem 1 (Kirchhoff’s Matrix-Tree Theorem)** $\det(L_G[i])$ gives the number of spanning trees in $G$ (for any $i$).

In order to do the first proof, we need to use the following fact.

**Fact 1** Let $E_{ii}$ be a matrix with 1 in the $(i,i)^{th}$ entry and 0s elsewhere. Then

$$\det(A + E_{ii}) = \det(A) + \det(A[i]).$$

If you think about a determinant as being the sum over all permutations of the products of the entries corresponding to the permutation, the fact makes sense: we’ve increased the $(i,i)$ entry, $a_{ii}$, to $(a_{ii} + 1)$, and we can think about each permutation that uses the $(i,i)$ entry either multiplying by $a_{ii}$ (in which case we just get $\det(A)$) or by the 1, in which case, we get the sum over all the permutations that avoid the $i^{th}$ row and column, or $\det(A[i])$.

**Proof of Theorem [1]** Our first proof will be by induction on the number of vertices and edges of graph $G$.

**Base case**: $G$ is an empty graph of two vertices, then

$$L_G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so that $L_G[i] = [0]$ and $\det(L_G[i]) = 0$.

**Inductive step**: Suppose there exists $e = (i,j)$ incident in $i$. If there is not and $i$ is an isolated vertex, then there are zeros along $i^{th}$ row and column of $L_G$. Then $\det(L_G[i]) = \det(L_{G-e}) = 0 = \prod_{i=1}^{n} \lambda_i$ and, as we showed previously, $\lambda_1 = 0$ for any

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\(^{0}\)This lecture is derived from Cvetković, Rowlinson, and Simić, *An Introduction to the Theory of Graph Spectra*, Sections 7.1 and 7.2, and Godsil and Royle, *Algebraic Graph Theory*, Section 13.2.
$L_G$. Note also that the number of spanning trees is 0 if $i$ is isolated, so the theorem holds in this case.

Now we introduce some notations. Let $\tau(G)$ is the number of spanning trees in $G$, let $G - e$ be $G$ with edge $e$ removed, and $G/e$ be $G$ with edge $e$ contracted. See below for an illustration of graph contraction.

![Illustration of graph contraction](image)

Before contraction  
After contraction

For any spanning tree $T$, either $e \in T$ or $e \not\in T$. We note that $\tau(G/e)$ gives the number of trees $T$ with $e \in T$, while $\tau(G - e)$ gives the number of trees $T$ with $e \not\in T$. Thus

$$\tau(G) = \tau(G\backslash e) + \tau(G - e);$$

note that the first term is $G$ with one fewer edge, while the second has one fewer vertex, and so these will serve as the basis of our induction.

First we try to relate $L_G$ to $L_{G-e}$, and we observe that $L_G[i] = L_{G-e}[i] + E_{jj}$ (that is, if we remove edge $e$, then the only difference in the matrix $L_G[i]$ is that we have to correct for the change in degree of $j$). Then by the Fact 1

$$\det(L_G[i]) = \det(L_{G-e}[i]) + \det(L_{G/e}[j]) = \tau(G - e) + \tau(G/e) = \tau(G).$$

where the second equation follows by induction; this completes the proof.

For the second proof of the theorem, we need the following fact which explains how to take the determinant of the product of rectangular matrices.
Fact 2 (Cauchy-Binet Formula) Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times n}$, for $m \geq n$. Let $A_S$ (respectively $B_S$) be submatrices formed by taking the columns (respectively rows) indexed by $S \subseteq [m]$ of $A$ (respectively $B$).

Let $\binom{[m]}{n}$ be the set of all size $n$ subsets of $[m]$. Then

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_S) \det(B_S).$$

Recall that $L_G = \sum_{(i,j) \in E} (e_i - e_j)(e_i - e_j)^T$. Thus we can write $L_G = BB^T$ where $B \in \mathbb{R}^{m \times n}$ has one column of $B$ per edge $(i, j)$, with the column $(e_i - e_j)$. Since we can write $L_G = BB^T$, this is yet another proof that $L_G$ is positive semidefinite. Then if $B[i]$ denotes $B$ with its $i^{th}$ row omitted, then $L_G[i] = B[i]B[i]^T$. We let $B_S[i]$ denote $B[i]$ with just the columns of $S \subseteq E$.

We need the following lemma, whose proof we defer for a moment.

Lemma 2 For $S \subseteq E$, $|S| = n - 1$, $|\det(B_S[i])| = 1$ if $S$ is a spanning tree, $0$ otherwise.

The second proof of the matrix-tree theorem now becomes very short.

Proof of Theorem 1:

$$\det(L_G[i]) = \det(B[i]B[i]^T)$$

$$= \sum_{S \in \binom{E}{n-1}} (\det(B_S[i]))(\det(B_S[i]))$$

$$= \tau(G),$$

where the second equation follows by the Cauchy-Binet formula, and the third by Lemma 2.

We can now turn to the proof of the lemma.

Proof of Lemma 2: Assume that the edges in $B_S[i]$ are “directed” however we want; that is, we can change the column corresponding to $(i, j)$ from $e_i - e_j$ to $e_j - e_i$, since this only flips the sign of the determinant.

If $S \subseteq E$, $|S| = n - 1$, and $S$ is not a spanning tree, then it must contain a cycle. We direct edges around the cycles. If we then sum the columns of $B_S[i]$ corresponding to the cycle, we obtain the 0 vector, which implies that the columns of $B_S[i]$ are linearly dependent, and thus $\det(B_S[i]) = 0$.

Now we suppose that $S$ is a spanning tree; we prove the lemma statement by induction on $n$.

Base case $n = 2$. Then

$$B_S = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
so that $B_S[i] = \pm 1$, and thus $\det(B_S[i]) = 1$.

**Inductive case:** Suppose the lemma statement is true for graphs of size $n-1$. Let $j$ leaf of the tree $j \neq i$. Let $(k, j)$ be edge incident on $j$. We exchange rows/columns so that $(k, j)$ is last column, and $j$ is last row; this may flip sign of determinant, but that doesn’t matter. Then

$$B_S[i] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ & & & \end{bmatrix}$$

Thus if we expand the determinant along the last row we get

$$| \det(B_S[i]) | = | \det(B_S - \{(k,j)\}[i]) | = 1.$$ 

The last equality follows by induction since $S - \{(k, j)\}$ is a tree on the vertex set without $j$, since we assumed that $j$ is a leaf. \hfill \square

## 2 Consequences of the Matrix-Tree Theorem

Once we have the matrix-tree theorem, there are a number of interesting consequences, which we explore in this section. Given a square matrix $A \in \mathbb{R}^{n \times n}$, let $A_{ij}$ be matrix without row $i$ column $j$ (so $A[i] = A_{ii}$). Let $C_{ij} = (-1)^{i+j} \det(A_{ij})$ be the $i, j$ cofactor of $A$. Then we define the adjugate $\text{adj}(A)$ as the matrix with $i, j$ entry $C_{ji}$. We will need the following fact.

**Fact 3**

$$A \text{adj}(A) = \det(A)I.$$ 

By the matrix-tree theorem, the $(i, i)$ cofactor of $L_G$ is equal to $\tau(G)$. But we can say something even stronger.

**Theorem 3** Every cofactor of $L_G$ is $\tau(G)$, so that

$$\text{adj}(L_G) = \tau(G)J.$$ 

**Proof:**

If $G$ is not connected, then $\tau(G) = 0$ and $\lambda_2(L_G) = 0 = \lambda_1(L_G)$. So the rank of $L_G$ rank is at most $n - 2$. Then $\det((L_G)_{ij}) = 0$, which implies that $\text{adj}(L_G) = 0$, as desired.
If $G$ is connected, since $\det(L_G) = 0$, by the fact above $L_G \text{adj}(L_G) = 0$ (i.e. the zero matrix). Because $G$ is connected, multiples of $e$ are the only eigenvectors of $L_G$ with eigenvalue of 0. Thus every column of $\text{adj}(L_G)$ must be some multiple of $e$. But we know that for the $i$th column of $\text{adj}(L_G)$, its $i$th entry is $\tau(G)$, so the column itself must be $\tau(G)e$, and the lemma statement follows.

We conclude with one more theorem.

**Theorem 4** Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $L_G$. Then

$$\tau(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i.$$  

**Proof:** The theorem is true if $G$ is not connected, since then $\lambda_2 = 0$ and $\tau(G) = 0$.

Otherwise, we will look at linear term of the characteristic polynomial in two different ways. In the first way, the characteristic polynomial is

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = \lambda(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_n),$$

so the linear term is

$$(-1)^{n-1} \prod_{i=2}^{n} \lambda_i.$$ 

For the second way, we want the linear term of $\det(\lambda I - L_G)$; the matrix looks like the following:

$$\begin{pmatrix}
\lambda - d(1) & \cdots & -L_G \\
\vdots & \ddots & \vdots \\
-L_G & \cdots & \lambda - d(n)
\end{pmatrix}$$

If we think about the determinant as the sum over all permutations of the products of the entries corresponding to the permutation, then we get a linear term in $\lambda$ whenever an $(i, i)$ term is part of the permutation, but no other diagonal entries are part of the permutation; also, if the $(i, i)$ term is part of the permutation then no other entry from row and column $i$ is part of the permutation. Finally, since all the other entries are negations of their entry in $L_G$, we get that if we have a linear term in $\lambda$ because we include the $(i, i)$ term of the matrix as part of the permutation, the linear term is $(-1)^{n-1} \det(L_G[i])$. Summing over all $(i, i)$ entries, the linear term of $\lambda$ in $\det(\lambda I - L_G)$ is

$$(-1)^{n-1} \sum_{i=1}^{n} \det(L_G[i]) = (-1)^{n-1} \cdot n \cdot \tau(G).$$

Thus we have that $\tau(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i.$