ORIE 6334 Spectral Graph Theory September 8, 2016

Lecture 6

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1 The Matrix-Tree Theorem

In this lecture, we continue to see the usefulness of the graph Laplacian, and its connection to yet another standard concept in graph theory, that of a spanning tree. Let $A[i]$ be the matrix A with its i^{th} column and row removed. We will give two different proofs of the following.

Theorem 1 (Kirchhoff's Matrix-Tree Theorem) $\det(L_G[i])$ *gives the number of spanning trees in* G *(for any* i*).*

In order to do the first proof, we need to use the following fact.

Fact 1 *Let* E_{ii} *be a matrix with 1 in the* $(i,i)^{th}$ *entry and 0s elsewhere. Then*

$$\det(A + E_{ii}) = \det(A) + \det(A[i]).$$

If you think about a determinant as being the sum over all permutations of the products of the entries corresponding to the permutation, the fact makes sense: we've increased the (i,i) entry, a_{ii}, to $(a_{ii}+1)$, and we can think about each permutation that uses the (i,i) entry either multiplying by a_{ii} (in which case we just get $\det(A)$ or by the 1, in which case, we get the sum over all the permutations that avoid the ith row and column, or $\det(A[i])$.

Proof of Theorem 1: Our first proof will be by induction on the number of vertices and edges of graph G.

Base case: G is an empty graph of two vertices, then

$$L_G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

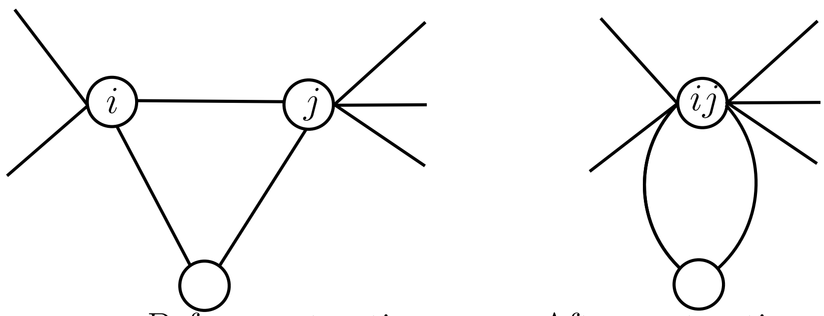
so that $L_G[i] = [0]$ and $\det(L_G[i]) = 0$.

Inductive step: Suppose there exists $e = (i,j)$ incident in i. If there is not and i is an isolated vertex, then there are zeros along i^{th} row and column of L_G. Then $\det(L_G[i]) = \det(L_{G-i}) = 0 = \prod_{i=1}^{n} \lambda_i$ and, as we showed previously, $\lambda_1 = 0$ for any

[0]This lecture is derived from Cvetković, Rowlinson, and Simić, *An Introduction to the Theory of Graph Spectra*, Sections 7.1 and 7.2, and Godsil and Royle, *Algebraic Graph Theory*, Section 13.2.

L_G. Note also that the number of spanning trees is 0 if i is isolated, so the theorem holds in this case.

Now we introduce some notations. Let $\tau(G)$ is the number of spanning trees in G, let $G - e$ be G with edge e removed, and G/e be G with edge e contracted. See below for an illustration of graph contraction.



Before contraction After contraction

For any spanning tree T, either $e \in T$ or $e \notin T$. We note that $\tau(G/e)$ gives the number of trees T with $e \in T$, while $\tau(G-e)$ gives the number of trees T with $e \notin T$. Thus

$$\tau(G) = \tau(G \backslash e) + \tau(G - e);$$

note that the first term is G with one fewer edge, while the second has one fewer vertex, and so these will serve as the basis of our induction.

First we try to relate L_G to L_{G-e}, and we observe that $L_G[i] = L_{G-e}[i] + E_{jj}$ (that is, if we remove edge e, then the only difference in the matrix $L_G[i]$ is that we have to correct for the change in degree of j). Then by the Fact 1

$$\begin{aligned} \det(L_G[i]) &= \det(L_{G-e} + E_{jj}) \\ &= \det(L_{G-e}[i]) + \det(L_{G-e}[i,j]) \\ &= \det(L_{G-e}[i]) + \det(L_G[i,j]), \end{aligned}$$

where by $L_G[i,j]$ we mean L_G with both the ith and jth rows and columns removed; the last equality follows since once we've removed both the ith and jth rows and columns there's no difference between L_G and L_{G-e} for $e = (i,j)$.

Now to relate L_G to $L_{G/e}$. Suppose we contract i onto j (so that $L_{G/e}$ has no row/column corresponding to i). Then $L_{G/e}[j] = L_G[i,j]$.

Thus we have that

$$\begin{aligned} \det(L_G[i]) &= \det(L_{G-e}[i]) + \det(L_{G/e}[j]) \\ &= \tau(G-e) + \tau(G/e) = \tau(G). \end{aligned}$$

where the second equation follows by induction; this completes the proof. □

For the second proof of the theorem, we need the following fact which explains how to take the determinant of the product of rectangular matrices.

Fact 2 (Cauchy-Binet Formula) *Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times n}$, for $m \geq n$. Let A_S (respectively B_S) be submatrices formed by taking the columns (respectively rows) indexed by $S \subseteq [m]$ of A (respectively B).*

Let $\binom{[m]}{n}$ be the set of all size n subsets of $[m]$. Then

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_S)\det(B_S).$$

Recall that $L_G = \sum_{(i,j) \in E}(e_i - e_j)(e_i - e_j)^T$. Thus we can write $L_G = BB^T$ where $B \in \mathbb{R}^{m \times n}$ has one column of B per edge (i,j), with the column $(e_i - e_j)$. Since we can write $L_G = BB^T$, this is yet another proof that L_G is positive semidefinite. Then if $B[i]$ denotes B with its i^{th} row omitted, then $L_G[i] = B[i]B[i]^T$. We let $B_S[i]$ denote $B[i]$ with just the columns of $S \subseteq E$.

We need the following lemma, whose proof we defer for a moment.

Lemma 2 *For $S \subseteq E$, $|S| = n - 1$, $|det(B_S[i])| = 1$ if S is a spanning tree, 0 otherwise.*

The second proof of the matrix-tree theorem now becomes very short.

Proof of Theorem 1:

$$\begin{aligned}
\det(L_G[i]) &= \det(B[i]B[i]^T) \\
&= \sum_{S \in \binom{E}{n-1}} (\det(B_S[i]))(\det(B_S[i])) \\
&= \tau(G),
\end{aligned}$$

where the second equation follows by the Cauchy-Binet formula, and the third by Lemma 2. □

We can now turn to the proof of the lemma.

Proof of Lemma 2: Assume that the edges in $B_S[i]$ are "directed" however we want; that is, we can change the column corresponding to (i,j) from $e_i - e_j$ to $e_j - e_i$, since this only flips the sign of the determinant.

If $S \subseteq E$, $|S| = n - 1$, and S is not a spanning tree, then it must contain a cycle. We direct edges around the cycles. If we then sum the columns of $B_S[i]$ corresponding to the cycle, we obtain the 0 vector, which implies that the columns of $B_S[i]$ are linearly dependent, and thus $\det(B_S[i]) = 0$.

Now we suppose that S is a spanning tree; we prove the lemma statement by induction on n.

<u>**Base case**</u> $n = 2$. Then

$$B_S = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so that $B_S[i] = \pm 1$, and thus $\det(B_S[i] = 1)$.

Inductive case: Suppose the lemma statement is true for graphs of size $n-1$. Let j leaf of the tree $j \neq i$. Let (k,j) be edge incident on j. We exchange rows/columns so that (k,j) is last column, and j is last row; this may flip sign of determinant, but that doesn't matter. Then

$$B_S[i] = \begin{array}{c} \begin{array}{cccc} & & & (k,j) \end{array} \\ \left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & 1 \\ & & & 0 \\ \hline 0 & \dots & 0 & -1 \end{array}\right] \end{array}$$

Thus if we expand the determinant along the last row we get

$$|\det(B_S[i])| = |\det(B_{S-\{(k,j)\}}[i])| = 1.$$

The last equality follows by induction since $S - \{(k,j)\}$ is a tree on the vertex set without j, since we assumed that j is a leaf. □

2 Consequences of the Matrix-Tree Theorem

Once we have the matrix-tree theorem, there are a number of interesting consequences, which we explore in this section. Given a square matrix $A \in \mathbb{R}^{n \times n}$, let A_{ij} be matrix without row i column j (so $A[i] = A_{ii}$). Let $C_{ij} = (-1)^{i+j}\det(A_{ij})$ be the i,j *cofactor* of A. Then we define the *adjugate* $\operatorname{adj}(A)$ as the matrix with i,j entry C_{ji}. We will need the following fact.

Fact 3

$$A \operatorname{adj}(A) = \det(A) I.$$

By the matrix-tree theorem, the (i,i) cofactor of L_G is equal to $\tau(G)$. But we can say something even stronger.

Theorem 3 *Every cofactor of L_G is $\tau(G)$, so that*

$$\operatorname{adj}(L_G) = \tau(G) J.$$

Proof:

If G is not connected, then $\tau(G) = 0$ and $\lambda_2(L_G) = 0 = \lambda_1(L_G)$. So the rank of L_G rank is at most $n-2$. Then $\det((L_G)_{ij}) = 0$, which implies that $\operatorname{adj}(L_G) = 0$, as desired.

If G is connected, since $\det(L_G) = 0$, by the fact above $L_G \operatorname{adj}(L_G) = 0$ (i.e. the zero matrix). Because G is connected, multiples of e are the only eigenvectors of L_G with eigenvalue of 0. Thus every column of $\operatorname{adj}(L_G)$ must be some multiple of e. But we know that for the ith column of $\operatorname{adj}(L_G)$, its ith entry is $\tau(G)$, so the column itself must be $\tau(G)e$, and the lemma statement follows. □

We conclude with one more theorem.

Theorem 4 *Let $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the eigenvalues of L_G. Then*

$$\tau(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i.$$

Proof: The theorem is true if G is not connected, since then $\lambda_2 = 0$ and $\tau(G) = 0$.

Otherwise, we will look at linear term of the characteristic polynomial in two different ways. In the first way, the characteristic polynomial is

$$(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) = \lambda(\lambda - \lambda_2)(\lambda - \lambda_3)...(\lambda - \lambda_n),$$

so the linear term is

$$(-1)^{n-1} \prod_{i=2}^{n} \lambda_i.$$

For the second way, we want the linear term of $\det(\lambda I - L_G)$; the matrix looks like the following:

$$\begin{pmatrix} {\scriptstyle \lambda - d(1)} & & -L_G & \\ & \ddots & & \\ -L_G & & \ddots & \\ & & & {\scriptstyle \lambda - d(n)} \end{pmatrix}$$

If we think about the determinant as the sum over all permutations of the products of the entries corresponding to the permutation, then we get a linear term in λ whenever an (i, i) term is part of the permutation, but no other diagonal entries are part of the permutation; also, if the (i, i) term is part of the permutation then no other entry from row and column i is part of the permutation. Finally, since all the other entries are negations of their entry in L_G, we get that if we have a linear term in λ because we include the (i, i) term of the matrix as part of the permutation, the linear term is $(-1)^{n-1} \det(L_G[i])$. Summing over all (i, i) entries, the linear term of λ in $\det(\lambda I - L_G)$ is

$$(-1)^{n-1} \sum_{i=1}^{n} \det(L_G[i]) = (-1)^{n-1} \cdot n \cdot \tau(G).$$

Thus we have that $\tau(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i$.

□