ORIE 6334 Bridging Continuous and Discrete Optimization Sept 9, 2019

Lecture 2

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1 Eigenvalue Interlacing Theorem

The following theorem is known as the *eigenvalue interlacing theorem.* Today we will see several implications of this theorem, including a brand-new result proving the *sensitivity conjecture* from complexity theory.

Theorem 1 (Eigenvalue Interlacing Theorem) *Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric. Let $B \in \mathbb{R}^{m\times m}$ with $m < n$ be a principal submatrix (obtained by deleting both i-th row and i-th column for some values of i). Suppose A has eigenvalues $\lambda_n \leq \lambda_{n-1} \cdots \leq \lambda_1$ and B has eigenvalues $\beta_m \leq \beta_{m-1} \cdots \leq \beta_1$. Then*

$$\lambda_{k+m-n} \leq \beta_k \leq \lambda_k \quad \textit{for} \quad k = 1, \cdots, m$$

And if $m = n - 1$,

$$\lambda_n \leq \beta_{n-1} \leq \lambda_{n-1} \leq \beta_{n-2} \leq \cdots \leq \beta_1 \leq \lambda_1$$

2 Clique and Chromatic Number

We now use the eigenvalue interlacing theorem to prove some statements about two particular graph quantities, the clique number and the chromatic number.

Definition 1 *The clique number of G, $\omega(G)$, is the size of the largest $S \subseteq V$ such that for all $i, j \in S$, $(i, j) \in E$.*

Definition 2 *The chromatic number $\chi(G)$ is the fewest number of colors needed such that we can assign one color to each vertex and for all $(i, j) \in E$, i, j are assigned different colors.*

Observation 1 *$\chi(G) \geq \omega(G)$.*

The observation follows since every vertex in the maximum clique needs to be assigned a different color: if two vertices in the clique are assigned the same color, then since there is an edge between them, the two endpoints of that edge are not assigned different colors.

Consider the complete graph on n nodes $G \equiv K_n$; that is, there is an edge between every pair of vertices. Then $\omega(G) = n = \chi(G)$. The adjacency matrix of G is $A = J - I$ where J is the matrix of all ones. Let

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then

$$Ae = (J - I)e = ne - e = (n-1)e$$

Therefore e is an eigenvector for eigenvalue $n-1$.

We can assume all remaining eigenvectors v satisfy $v \perp e$, i.e. $v^T e = 0$.

For any vector v such that $v^T e = 0$,

$$Av = (J - I)v = 0 - v = -v$$

This means any v such that $\langle e, v \rangle = 0$ is an eigenvector of eigenvalue -1. So we have eigenvalue -1 with multiplicity $n-1$.

Now consider an arbitrary graph G. Let A be its adjacency matrix, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues.

Claim 2 $\lambda_1 \geq \omega(G) - 1$.

Proof: Let B be the principal submatrix of A corresponding to the largest clique of G. Let m be the clique number, $m = \omega(G)$. Then, $B = J_m - I_m$, $\beta_1 = m - 1$. By the Interlacing Theorem, $\lambda_1 \geq \beta_1 = m - 1 = \omega(G) - 1$. □

We can in fact prove something slightly stronger. The following theorem strengthens that bound of the claim since $\omega(G) \leq \chi(G)$.

3 Wilf's Theorem

Theorem 3 (Wilf 1967) $\chi(G) \leq \lfloor \lambda_1 \rfloor + 1$

Before we can prove this we need a lemma. Let $d_G(i)$ be the degree of node i in G, $\Delta(G) = \max_{i \in V} d_G(i)$ and $d_{ave} = \frac{1}{n} \sum_{i \in V} d_G(i)$.

Lemma 4 $d_{ave} \leq \lambda_1 \leq \Delta$.

Observation 2 $\chi(G) \leq \Delta + 1$.

This is true because if we color the graph greedily, we will never get stuck: if we color a vertex, it has at most Δ neighbors that have already been colored, and so we can color it with the $(\Delta + 1)$st color. So we note that the lemma implies that Wilf's theorem is stronger than this greedy coloring result.

In order to prove the lemma, we first need the following fact, which we will prove later in the course.

Fact 1

$$\lambda_1 = \max_x \frac{x^T A x}{x^T x}.$$

Proof of Lemma 4:

Proof of $\lambda_1 \geq d_{ave}$:

$$\lambda_1 = \max_x \frac{x^T A x}{x^T x} \geq \frac{e^T A e}{e^T e} = \frac{\sum_{i,j} a_{ij}}{n} = \frac{\sum_{i \in V} d_G(i)}{n} = d_{ave}.$$

Proof of $\lambda_1 \leq d_{ave}$:

Let v be the eigenvector associated with λ_1, so that $Av = \lambda_1 v$. WLOG, assume $|v(1)| \geq |v(j)|\ \forall j$. Then,

$$\begin{aligned}
|\lambda_1 v(1)| &= |Av(1)| \\
&= |\sum_{j=1}^{n} a_{ij} v(j)| \\
&= |\sum_{j:\ (1,j)\in E} a_{1j} v(j)| \\
&\leq \sum_{j:\ (1,j)\in E} |a_{1j}||v(j)| \\
&\leq |v(1)||\sum_{j:\ (1,j)\in E} |a_{1j}| \\
&\leq |v(1)|\Delta(G).
\end{aligned}$$

Thus we get that $\lambda_1 \leq \Delta(G)$. □

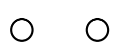
The following corollary will be useful when we get to the proof of the sensitivity conjecture.

Corollary 5 *It is still the case that* $\lambda_1 \leq \Delta(G)$ *if* $a_{ij} = 0$ *when* $(i,j) \notin E$ *and* $|a_{ij}| \leq 1\ \forall i, j$.

Proof of Wilf's Theorem: The proof is by induction on n.

Base case $n = 2$,

 $\lambda_1 = 1, \quad \chi_1(G) = 2$

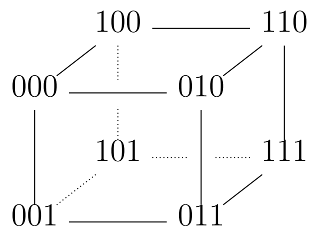
 $\lambda_1 = 0, \quad \chi_1(G) = 1$

Inductive step: Suppose the theorem holds on all graphs with $n-1$ vertices. Let G be a graph with n vertices. Since $d_{ave} \leq \lambda_1$, there must exists a vertex v of degree $\leq \lambda_1$. Remove this vertex v and call the resulting graph G'. Let B be its adjacency matrix and β_1 be its largest eigenvalue. By the Interlacing Theorem, $\beta_1 \leq \lambda_1$. By induction, we can color G' with $\lfloor \beta_1 \rfloor + 1 \leq \lfloor \lambda_1 \rfloor + 1$ colors. Since v has less than $\lfloor \lambda_1 \rfloor$ neighbors, we color v with one of the $\lfloor \lambda_1 \rfloor + 1$. □

4 Huang's theorem on sensitivity conjecture

We now turn to a very recent result. First we need some definitions.

Definition 3 *The hypercube graph* $Q_d = (V, E)$ *is defined by* $V = \{x\colon x \in \{0,1\}^d\}$ *and* $E = \{(x, y)\colon x, y$ *differ in exactly one bit*$\}$.
Example:



Q_3

Definition 4 $H = (V_H, V_E)$ *is an induced subgraph of* $G = (V, E)$ *if* $V_H \subseteq V$ *and* $E_H = \{(x, y)\colon x, y \in V_H$ *and* $(x, y) \in E\}$.

The following result is just a few months old. We'll later explain its connection to complexity theory.

Theorem 6 (Huang 7/1/2019) *For any* $d \geq 1$, *let* H *be an induced subgraph of* Q_d *such that* $|V_H| \geq 2^{d-1} + 1$. *Then,* $\Delta(H) \geq \sqrt{d}$.

We'll need the following facts, the first of which we showed last time, and the second we will show in a few lectures.

Fact 2 *If* λ *is an eigenvalue of* A, *then* λ^2 *is an eigenvalue of* A^2, *and* $Tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$.

Define $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_d = \begin{bmatrix} A_{d-1} & I \\ I & -A_{d-1} \end{bmatrix}$, where $A_d \in \mathbb{R}^{2^d \times 2^d}$.

Lemma 7 A_d *has an eigenvalue* $\sqrt{d}$ *of multiplicity* 2^{d-1}. $-\sqrt{d}$ *of multiplicity* 2^{d-1}.
Proof: We show by induction that $Ad^2 = dI$.
<u>Base case</u>: When $d = 1$, $A_1^2 = I$.
<u>Inductive Step</u>: Suppose $A_{d-1}^2 = (d-1)I$. Then

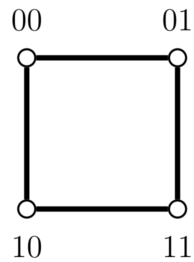
$$\begin{aligned} A_d^2 &= \begin{bmatrix} A_{d-1} & I \\ I & -A_{d-1} \end{bmatrix} \cdot \begin{bmatrix} A_{d-1} & I \\ I & -A_{d-1} \end{bmatrix} \\ &= \begin{bmatrix} A_{d-1}^2 + I & 0 \\ 0 & A_{d-1}^2 + I \end{bmatrix} \\ &= \begin{bmatrix} (d-1)I + I & 0 \\ 0 & (d-1)I + I \end{bmatrix} \\ &= dI. \end{aligned}$$

So all eigenvalues of A_d^2 are d. Then, $\sqrt{d}$ and $-\sqrt{d}$ must be the eigenvalues of A_d. Since $Tr(A_d) = 0$, we must have half of the eigenvalues are $\sqrt{d}$ and half of them are $-\sqrt{d}$. $\square$

Observation 3 *The absolute value of A_d (entrywise) is the adjacency matrix of Q_d.*
Example:

Q_1

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Q_2

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Proof of Huang's theorem: Let H be an induced subgraph of Q_d. Let A_H be the corresponding principal submatrix of A_d. Then, $\Delta(H) \geq \lambda(A_H)$, by the corollary we showed earlier. By the eigenvalue interlacing theorem,

$$\lambda_1(A_H) \geq \lambda_{1+2^d-|V_H|}(A_d) = \lambda_k(A_d) \geq \sqrt{d},$$

for $k \leq 2^{d-1}$.

$\square$

This result is important since it proved the sensitivity conjecture. We will now state what this is. For $x \in \{0,1\}^d$, $S \subseteq \{1, \cdots, d\}$. Let x^S be x with bits in positions in S flipped. Define boolean function $f : \{0,1\}^d \to \{0,1\}$.

Definition 5 *The local sensitivity $s(f,x)$ is defined as the number of indices i such that $f(x) \neq f(x^i)$, i.e,*

$$s(f,x) = |\{i : f(x) \neq f(x^i)\}|$$

Definition 6 *The sensitivity $s(f)$ of f is $\max_x s(f,x)$.*

Definition 7 *The local block sensitivity $bs(f,x)$ is the maximum number of subpartitions $B_1, \cdots, B_k$ such that $f(x) \neq f(x^{B_i})$ for each i.*

Definition 8 *The block sensitivity $bs(f)$ of f is $\max_x bs(f,x)$.*

Note that $bf(f) \geq s(f)$. The sensitivity conjecture is as follows; it states that the sensitivity and block sensitivity are polynomially related for all boolean functions.

Conjecture 1 (Nisan, Szegedy '92) *$\exists$ constant $k > 0$ such that for all boolean f,*

$$bs(f) \leq (s(f))^k.$$

Let $deg(f)$ be the degree of the unique multilinear real polynomial of the function f. Then Nisan and Szegedy also showed the following.

Theorem 8 (Nisan, Szegedy '92) *$bs(f) \leq 2deg^2(f)$.*

Finally, the connection between the graph theoretic problem studied by Huang and the sensitivity conjecture was made by Gotsman and Linial.

Theorem 9 (Gotsman and Linial '92) *For any monotone function $h : \mathbb{N} \to \mathbb{R}$, the following are equivalent:*

- *For any induced subgraph H of Q_d such that $|V_H| \neq 2^{d-1}$, with H' the induced subgraph on $V - V_H$, $\max(\Delta(H), \Delta(H')) \geq h(d)$.*
- *For any boolean f, $s(f) \geq h(deg(f))$.*

Thus Huang's theorem implies that for $h(d) = \sqrt{d}$, $s(f) \geq \sqrt{deg(f)}$. Then we have that

$$s(f) \leq bs(f) \leq 2deg^2(f) \leq 2s(f)^4,$$

proving the conjecture for $k = 4$.