Lecture 13

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1 Electrical flow

Consider an undirected graph $G = (V, E)$ such that each edge (i, j) has resistance $r(i, j)$ (or conductance $c(i, j) = 1/r(i, j)$). A *current flow* $f(i, j)$ is one that obeys both

- Kirchoff's current law (KCL) (i.e. flow conservation)

$$\text{flow into node } i = \text{flow leaving node } i$$

- Ohm's Law: there exist potentials $p(i)$, for all $i \in V$, such that

$$f(i, j) = \frac{p(i) - p(j)}{r(i, j)} = c(i, j)(p(i) - p(j)).$$

  Then notice that if Ohm's Law is obeyed, then it must be the case that

$$f(j, i) = \frac{p(j) - p(i)}{r(i, j)} = -\frac{p(i) - p(j)}{r(i, j)} = -f(i, j),$$

  so the flow from j to i is the negative of the flow from i to j. We call this condition *skew symmetry.*

Since G is undirected, we'll try use (i, j) when we want direction (e.g. $f(i, j)$ indicates the flow from i to j), and use $\{i, j\}$ when we don't (e.g. summing over $\{i, j\} \in E$). Notice there are also some exceptions, say, resistance $r(i, j)$, which is inherently undirected, and we assume this is clear from the context. It will also be useful to assume each edge is oriented arbitrary; we denote this set of edges as $\vec{E}$.

Notice that it is very easy to find a current flow and potentials obeying KCL and Ohm's Law, i.e. a feasible solution: $f = 0$ and $p = 0$. To make things more interesting, we can think about supplying and demanding current from the circuit; let $b(i)$ be current supplied to i, where $b(i) > 0$ if it is a supply, and $b(i) < 0$ if it is a demand. Then

$$\begin{aligned} b(i) &= \sum_{j:\{i,j\} \in E} f(i, j), && \text{by KCL} \\ &= \sum_{j:\{i,j\} \in E} c(i, j)(p(i) - p(j)), && \text{by Ohm's Law.} \end{aligned}$$

[0]Based on the 2017 lecture slides.

If $b(s) = 1, b(t) = -1$ for some $s, t \in V$, we say f is an *s-t electrical flow*. An example is shown in Figure 1.

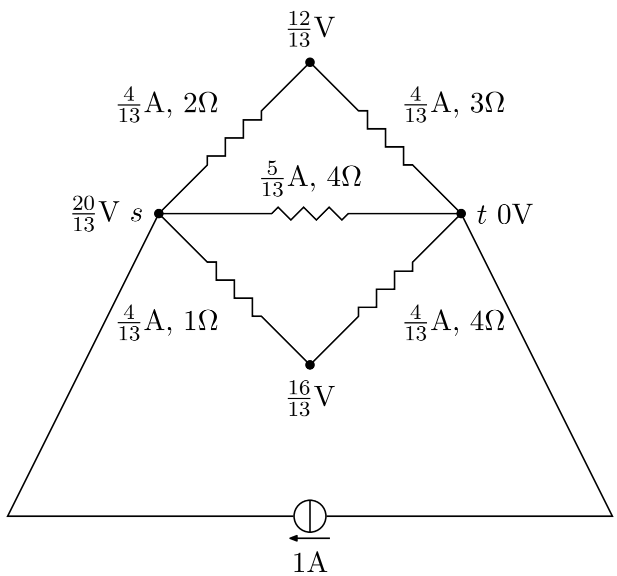

Figure 1: Example of a flow in an electrical network. We put in one unit of current (in amps) at s and remove one at t. The nodes show the potentials (in volts), and each edge has the current passing through it, followed by its resistance.

There is another law called Kirchoff's potential/voltage law (KPL or KVL):

- Kirchoff's potential law (KPL):

$$\sum_{(i,j)\in C} f(i,j)r(i,j) = 0 \qquad \text{for any directed cycle } C.$$

We prove that KPL is equivalent to Ohm's Law. It is occasionally useful for us to assume that the current flow is defined by KCL and KPL rather than KCL and Ohm's Law.

Lemma 1 *Ohm's Law is equivalent to KPL.*

Proof: If there exist potentials satisfying Ohm's Law, then for any directed cycle C,

$$\sum_{(i,j)\in C} f(i,j)r(i,j) = \sum_{(i,j)\in C} (p(i) - p(j)) = 0.$$

For the other direction, pick some spanning tree T with root r. Let P_i be the directed path in T from i to r. Then we can create the *tree-defined* potentials:

$$p(i) = \sum_{(k,l)\in P_i} f(k,l)r(k,l) \qquad \forall i \in V.$$

Notice that these are exactly the potentials we get by assuming that Ohm's Law is obeyed for each edge $\{i, j\} \in T$.

For any edge $(i, j) \in E - T$, let P_j^R be the r-i path in T (the superscript R denotes the reverse). Then let C be the directed cycle formed by the directed path P_i, the P_j^R, and the arc (j, i). Then we have that

$$\begin{aligned} p(i) - p(j) &= \sum_{(k,l) \in P_i} f(k,l)r(k,l) - \sum_{(k,l) \in P_j} f(i,l)r(k,l) \\ &= \sum_{(k,l) \in P_i \cup P_j^R} f(k,l)r(k,l) \\ &= \sum_{(k,l) \in C} f(k,l)r(k,l) - f(j,i)r(j,i) \\ &= 0 - f(j,i)r(j,i) = f(i,j)r(i,j), \end{aligned}$$

where the last equality holds by skew symmetry. □

The above proof implies the following corollary.

Corollary 2 *For any directed cycle C and tree-defined potentials p,*

$$r(i,j)f(i,j) - (p(i) - p(j)) = \sum_{(k,l) \in C} f(k,l)r(k,l).$$

This is all very interesting, but what does it have to do with spectral graph theory? Suppose that for a given supply vector b, we want to find the corresponding potentials p for the resulting electrical flow. Consider the weighted Laplacian with conductances as weights, i.e.

$$L_G = \sum_{\{i,j\} \in E} c(i,j)(e_i - e_j)(e_i - e_j)^T.$$

We claim that the potentials we want actually satisfies $L_G p = b$.

Claim 3 *$L_G p = b$.*

Proof: Note that

$$\begin{aligned} L_G p &= \sum_{\{i,j\} \in E} c(i,j)(e_i - e_j)(e_i - e_j)^T p \\ &= \sum_{\{i,j\} \in E} c(i,j)(e_i - e_j)(p_i - p_j), \end{aligned}$$

so that by flow conservation

$$(L_G p)(i) = \sum_{j:\{i,j\} \in E} c(i,j)(p(i) - p(j)) = \sum_{j:\{i,j\} \in E} f(i,j) = b(i).$$

Hence, it follows that $L_G p = b$. $\square$

Thus we get the potentials p by solving $L_G p = b$ for p. However, notice that L_G is singular (its smallest eigenvalue $\lambda_1 = 0$), and we cannot use L_G^{-1} directly. But we can use *pseudoinverse* instead. Recall

$$L_G^\dagger = \sum_{i:\lambda_i \neq 0} \frac{1}{\lambda_i} x_i x_i^T,$$

where $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues and $x_1, \ldots, x_n$ the associated orthonormal eigenvectors.

Claim 4 *$p = L_G^\dagger b$.*

Proof: In order for there to be a solution for p, we need $b^T e = 0$ (we assume G is connected $\Rightarrow \lambda_2 > 0$); that is, b is orthogonal to the null space of L_G. This is a natural physical condition; it says that the the amount of current supplied must be equal to the amount of current demanded, and there is overall conservation of current.

Assume that the eigenvectors $x_1, \ldots, x_n$ form an orthonormal basis with $x_1 = e/\sqrt{n}$; then we can write $b = \sum_2^n \alpha_i x_i$, since $b^T x_1 = 0$. Then

$$\begin{aligned} L_G L_G^\dagger b &= \left(\sum_{i=2}^n \lambda_i x_i x_i^T\right)\left(\sum_{i=2}^n \frac{1}{\lambda_i} x_i x_i^T\right) b \\ &= \left(\sum_{i=2}^n x_i x_i^T\right)\left(\sum_{i=2}^n \alpha_i x_i\right) = \sum_{i=2}^n \alpha_i x_i = b. \end{aligned}$$

Since $L_G p = b = L_G L_G^\dagger b$ we conclude that $L_G^\dagger b = p$. $\square$

Note that if G is connected, $\mathrm{tr}(L_G L_G^\dagger) = n - 1$ since $\left(\sum_{j=2}^n x_j x_j^T\right) x_i = x_i$ for $i = 2, \ldots, n$, and 0 for $i = 1$. So the spectrum of $L_G L_G^\dagger$ is 0 with multiplicity 1, and 1 with multiplicity $n - 1$.

The main reason this topic is of interest to us (and the main reason we have this course) is the following theorem shown by Spielman and Teng about a decade ago.

Theorem 5 (Spielman and Teng, 2004) *$L_G p = b$ can be solved for p (approximately) in $\tilde{O}(m)$ time.*

The significance of the paper is that we can solve this linear system in time nearly linear in the number of edges of the graph (i.e. essentially the number of nonzeroes of the Laplacian); this is useful since in many cases graphs really are sparse. There has since been a significant amount of followup work improving this result and finding various applications of it. We'll see some in upcoming lectures.

2 Effective Resistance and Energy

We now introduce the definitions for effective resistance and energy.

- The *effective resistance* $r_{\text{eff}}(i,j)$ between i and j is the potential drop between i and j induced by an i-j electrical flow. Essentially this quantity is the resistance between i and j if we replace the entire network by a single resistor. Since the potentials induced by an i-j electrical flow is the p such that $L_G p = e_i - e_j$ and $p(i) - p(j) = f(i,j)r(i,j)$ (i.e. $p = L_G^\dagger(e_i - e_j)$), the effective resistance is

$$r_{\text{eff}}(i,j) = p(i) - p(j) = (e_i - e_j)^T p = (e_i - e_j)^T L_G^\dagger (e_i - e_j).$$

- Given current f, the *energy* dissipated by a resistance r is $f^2 r$. Thus, the energy $\mathcal{E}(f)$ dissipated by our electrical network G with current flow f is

$$\begin{aligned}\mathcal{E}(f) &= \sum_{\{i,j\}\in E} f^2(i,j)r(i,j) = \sum_{\{i,j\}\in E} \frac{(p(i)-p(j))^2}{r(i,j)} \\ &= \sum_{\{i,j\}\in E} c(i,j)(p(i)-p(j))^2 = p^T L_G p, \text{ for associated potentials } p.\end{aligned}$$

  For an s-t electrical flow, $b = e_s - e_t$. If p is the associated potential with $L_G p = e_s - e_t$, we have that

$$\mathcal{E}(f) = p^T L_G p = p^T(e_s - e_t) = p(s) - p(t) = r_{\text{eff}}(s,t).$$

  Notice that this equivalence makes intuitive sense since the effective resistance is supposed to be the resistance of the entire network when we put a unit flow between s and t.

We end by showing flows, potentials and energy are actually closely related with each other.

Lemma 6 *The electrical flow f is the unique minimizer of energy $\mathcal{E}(f)$ of system among all flows g satisfying KCL for demand vector b.*

Proof: Let g be a flow different from f that satisfies KCL with demand b. Set $h = g - f$, then for any $i \in V$,

$$\sum_{j:\{i,j\}\in E} h(i,j) = \sum_{j:\{i,j\}\in E} g(i,j) - \sum_{j:\{i,j\}\in E} f(i,j) = 0$$

by flow conservation (KCL). Then

$$
\begin{aligned}
\mathcal{E}(g) &= \sum_{(i,j)\in\vec{E}} g(i,j)^2 r(i,j) \\
&= \sum_{(i,j)\in\vec{E}} (f(i,j)+h(i,j))^2 r(i,j) \\
&= \sum_{(i,j)\in\vec{E}} f(i,j)^2 r(i,j) + 2\sum_{(i,j)\in\vec{E}} f(i,j)h(i,j)r(i,j) + \sum_{(i,j)\in\vec{E}} h(i,j)^2 r(i,j) \\
&> \mathcal{E}(f) + 2\sum_{(i,j)\in\vec{E}} f(i,j)h(i,j)r(i,j) + 0 \qquad \text{(Since } f \neq g\text{)} \\
&= \mathcal{E}(f) + 2\sum_{(i,j)\in\vec{E}} (p(i)-p(j))h(i,j) \\
&= \mathcal{E}(f) + 2\sum_{i\in V} p(i) \sum_{j:\{i,j\}\in E} h(i,j) \qquad \text{(By skew-symmetry of } h\text{)} \\
&= \mathcal{E}(f).
\end{aligned}
$$

Therefore, we conclude that f is the unique minimizer of $\mathcal{E}(f)$. □

Lemma 7 *For a given b such that $b^T e = 0$, the potentials p for electrical flow f determined by b maximize $2x^T b - x^T L_G x$ over all $x \in \mathbb{R}^n$.*

Proof: Use calculus by setting $\nabla(2x^T b - x^T L_G x) = 2(b - L_G x) = 0$, i.e. $L_G x = b$. Thus, it directly follows that the potential p is the maximizer. □

Notice that by substituting x with optimal solution p,

$$2p^T b - p^T L_G p = 2p^T L_G p - p^T L_G p = p^T L_G p = \mathcal{E}(f).$$

In fact, the above two lemmas can be viewed as *dual* to each other, i.e. the primal and dual problems share the same optimal value, with flows and potential as their corresponding minimizer and maximizer respectively. We will use this fact in an upcoming algorithm.