Problem Set 3

Due Date: November 3, 2016

As a reminder, the collaboration policy from the syllabus is as follows:

> Your work on problem sets and exams should be your own. You may discuss approaches to problems with other students, but as a general guideline, such discussions may not involve taking notes. You must write up solutions on your own independently, and acknowledge anyone with whom you discussed the problem by writing their names on your problem set. You may not use papers or books or other sources (e.g. material from the web) to help obtain your solution.

Problem sets are due in class at the start of the lecture. If you would like an extension, please arrange it in advance.

1. In class, we argued that the random walk for a d-regular graph is ergodic if and only if the graph is connected and not bipartite. We indicated how the proof could be adapted to non-regular graphs. Complete the argument.

2. Let $\mathcal{E}(f)$ be the energy of s-t electrical flow f in a graph G. Let p be the associated potentials. Recall that $\mathcal{E}(f) = p^T L_G p$, and that $p(s) - p(t) = r_{\text{eff}}(s,t)$. Prove that for all vectors $x \in \Re^n$ for which $x(s) - x(t) = r_{\text{eff}}(s,t)$,

$$p^T L_G p \leq x^T L_G x.$$

   That is, the potentials p minimize the quadratic form $x^T L_G x$ among all vectors such that the potential difference $x(s) - x(t) = r_{\text{eff}}(s,t)$.

3. (Rayleigh's Monotonicity Principle) Given a graph G, let $\mathcal{E}(f,r)$ be the energy of a flow for supply vector b under resistances r. Let f be the electrical flow for supply vector b under resistances r, and let f' be the electrical flow for the same supply vector b under resistances $r' \geq r$. Prove that $\mathcal{E}(f',r') \geq \mathcal{E}(f,r)$.

4. Let G be an electrical network with resistances r. Prove that effective resistances obey the triangle inequality; that is, for any i, j, k,

$$r_{\text{eff}}(i,k) \leq r_{\text{eff}}(i,j) + r_{\text{eff}}(j,k).$$

5. Recall that in class we defined $A \preceq B$ iff $B - A \succeq 0$ (that is, $B - A$ is a positive semidefinite matrix).

Recall also that for a symmetric matrix A with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and orthonormal eigenvectors $x_1, \ldots, x_n$, we can express $A = XDX^T$ for X with x_i as its ith column, and D a diagonal matrix with $d_{ii} = \lambda_i$. For a function $f : \Re \to \Re$, define the spectral mapping $f(A) = X f(D) X^T$, where $f(D)$ is the diagonal matrix whose (i, i)th entry is $f(d_{ii})$.

(a) (Weyl's Monotonicity Theorem) Let A and B be symmetric $n \times n$ matrices such that $A \preceq B$. Prove that $\lambda_i(A) \leq \lambda_i(B)$ for all $1 \leq i \leq n$.

(b) Suppose that for $f : \Re \to \Re$ and $g : \Re \to \Re$, $f(x) \leq g(x)$ for all $x \in [\ell, u]$. Further suppose that $\lambda_i \in [\ell, u]$ for all eigenvalues λ_i of a symmetric matrix A. Then show that $f(A) \preceq g(A)$.

(c) Prove that for any spanning tree T of a connected, unweighted graph G,

$$L_T \preceq L_G \preceq st_T(G)L_T,$$

where $st_T(G)$ is the stretch of tree T for graph G.

(d) Let $e = (i, j)$ and $L_e = (e_i - e_j)(e_i - e_j)^T$. Assume that G is an unweighted graph. Prove that
$$L_e \preceq r_{\text{eff}}(i, j)L_G.$$