

Problem Set 1

Due Date: September 20, 2016

As a reminder, the collaboration policy from the syllabus is as follows:

Your work on problem sets and exams should be your own. You may discuss approaches to problems with other students, but as a general guideline, such discussions may not involve taking notes. You must write up solutions on your own independently, and acknowledge anyone with whom you discussed the problem by writing their names on your problem set. You may not use papers or books or other sources (e.g. material from the web) to help obtain your solution.

1. In class, we proved that for $A \in \mathbb{R}^{n \times n}$, A symmetric, (real) eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$, and orthonormal eigenvectors x_1, \dots, x_n , that

$$\lambda_{k+1} = \min_{x \perp \text{span}(x_1, \dots, x_k)} \frac{x^T A x}{x^T x}.$$

We asserted that it was also the case that

$$\begin{aligned} \lambda_{k+1} &= \min_{x \in \text{span}(x_{k+1}, \dots, x_n)} \frac{x^T A x}{x^T x} \\ &= \max_{x \perp \text{span}(x_{k+2}, \dots, x_n)} \frac{x^T A x}{x^T x} \\ &= \max_{x \in \text{span}(x_1, \dots, x_{k+1})} \frac{x^T A x}{x^T x}. \end{aligned}$$

Prove that the assertion is true.

2. One problem with the characterization of eigenvalues of the previous problem is that it requires us to know the eigenvectors x_1, \dots, x_k (or x_{k+2}, \dots, x_n) in order to compute λ_{k+1} . The *Courant-Fischer theorem* gives us a more general way of computing these eigenvalues. Prove that the following is true:

$$\begin{aligned} \lambda_{k+1} &= \min_{W \subseteq \mathbb{R}^n: \dim(W)=k+1} \max_{x \in W} \frac{x^T A x}{x^T x} \\ &= \max_{W \subseteq \mathbb{R}^n: \dim(W)=n-k} \min_{x \in W} \frac{x^T A x}{x^T x}. \end{aligned}$$

3. Suppose that G is connected and Δ is the maximum degree of any node in G . Let λ_1 be the maximum eigenvalue of the adjacency matrix A of G . Prove that G is Δ -regular if and only if $\lambda_1 = \Delta$.

4. Let $\lambda_n(M) \leq \lambda_{n-1}(M) \leq \dots \leq \lambda_1(M)$ be the eigenvalues of any matrix M . The *Courant-Weyl* inequalities state that for symmetric real matrices A and B ,

$$\lambda_i(A + B) \leq \lambda_j(A) + \lambda_{i-j+1}(B)$$

for $1 \leq j \leq i \leq n$ and

$$\lambda_i(A + B) \geq \lambda_j(A) + \lambda_{i-j+n}(B)$$

for $1 \leq i \leq j \leq n$.

- (a) Prove the inequalities. (Hint: recall the proof of the interlacing theorem. Now we need to think about three different vector spaces, for $A + B$, A , and B).
- (b) Use the inequalities to prove a type of interlacing theorem for Laplacians. Consider two graphs on the same vertex set, $G = (V, E)$ and $H = (V, E')$ in which $E' = E - \{e\}$ for a single edge $e \in E$. As usual, we assume that the eigenvalues of Laplacians are ordered as $\lambda_1(L_G) \leq \dots \leq \lambda_n(L_G)$. Then prove that

$$0 = \lambda_1(L_H) = \lambda_1(L_G) \leq \lambda_2(L_H) \leq \lambda_2(L_G) \leq \dots \leq \lambda_n(L_H) \leq \lambda_n(L_G).$$

(Aside: one cute application of this theorem is to show that the Petersen graph is not Hamiltonian by showing that these inequalities are violated by the spectrum of the Petersen graph and the spectrum of a cycle on 10 vertices (C_{10}), so that C_{10} is not a subgraph of the Petersen graph.)

5. Recall that in proving a lower bound on the clique number, $\omega(G)$, we used the following result of Motzkin and Straus:

$$\begin{aligned} \frac{1}{2} \left(1 - \frac{1}{\omega(G)} \right) = & \text{Maximize} \quad \sum_{(i,j) \in E} x_i x_j \\ & \text{subject to:} \\ & \sum_{i \in V} x_i \leq 1 \\ & x_i \geq 0 \quad \forall i \in V. \end{aligned}$$

Prove the result.