ORIE 6334 Spectral Graph Theory	September 6, 2016
Lecture 5	
Lecturer: David P. Williamson	*Scribe: Rahmtin Rotabi*

1 Graph Laplacians

Let's let $e_i \in \{0,1\}^n$ be the standard basis vectors (1 in the i-th coordinate, 0's else where).

A *Laplacian* of an undirected graph $G = (V, E)$,

$$L_G = \sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T.$$

Each term $(e_i - e_j)(e_i - e_j)^T$ is an $|V| \times |V|$ matrix that has $+1$ in the (i,i) and (j,j) coordinate, -1 in the (i,j) and (j,i) coordinate and the rest of the entries are all zero. Now, we define the following notation:

- $d(i)$: degree of i in G.
- D: $diag(d(i))$ is the $|V| \times |V|$ diagonal matrix where $D(i,i) = d(i)$.
- A: Adjacency matrix of graph A.

With this notation we can write $L_G = D - A$.

If G has weights $w(i,j), \forall (i,j) \in E$, then the *weighted Laplacian*,

$$L_G = \sum_{(i,j)\in E} w(i,j)(e_i - e_j)(e_i - e_j)^T.$$

Define $W = (w(i,j)) \in \Re^{n\times n}$ where $w(i,j) = 0$ if $(i,j) \notin E$ and $D = diag(d(i))$, where $d(i) = \sum_{(i,j)\in} w(i,j)$. Then $L_G = D - W$. We will sometimes denote this matrix by $L_{G,w}$.

An interesting and useful fact is that the Laplacian L_G is positive semidefinite. Let's briefly remember what this means, as well as some useful facts about such matrices.

Definition 1 *A matrix $A \in \Re^{n\times n}$ is positive semidefinite, if $x^T Ax \geq 0$ for all $x \in \Re^n$. If A is positive semidefinite we write $A \succeq 0$.*

[0]This lecture is derived from Lau's 2015 lecture notes, Lecture 2 (`https://cs.uwaterloo.ca/~lapchi/cs798/notes/L02.pdf`), Cvetković, Rowlinson, and Simić, *An Introduction to the Theory of Graph Spectra*, Section 7.4, and Mohar and Poljak, *Eigenvalues in Combinatorial Optimization*, Sections 2.1 and 2.4.

Given what we know about matrices, the following fact is easy to prove, but we will skip its proof.

Fact 1 *For a symmetric matrix A the following are equivalent:*

(i) $A \succeq 0$.

(ii) $A = VV^T$ *for some matrix* V.

(iii) A *has all non-negative eigenvalues.*

We can now show that L_G is positive semidefinite, which we will do in two different ways.

Claim 1 $L_G \succeq 0$.

Proof:
First proof:
Note L_G is symmetric.
We observe that if $A \succeq 0$ and $B \succeq 0$ then $A + B \succeq 0$, since

$$x^T(A + B)x = x^t Ax + x^t Bx \geq 0$$

for all $x \in \Re^n$. Note that by (ii), $(e_i - e_j)(e_i - e_j)^T \succeq 0$. So, by summing up all these terms we will get L_G and based on the observation above we can say $L_G \succeq 0$. $\Box$
Second proof:
Also we know that for any $x \in \Re^n$,

$$\begin{aligned} x^T L_G x &= x^T \left(\sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T \right) x \\ &= \sum_{(i,j)\in E} x^T (e_i - e_j)(e_i - e_j)^T x \\ &= \sum_{(i,j)\in E} (x(i) - x(j))(x(i) - x(j)) \\ &= \sum_{(i,j)\in E} (x(i) - x(j))^2 \qquad \geq 0. \end{aligned}$$

$\Box$

We will usually write the eigenvalues of L_G, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and since we know that L_G is positive semi-definite we can write $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

What is the spectrum of L_G? We observe that e (all 1s vector) is an eigenvector of eigenvalue 0 for L_G, since:

$$L_G e = \sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T e = \sum_{(i,j)\in E} (e_i - e_j \cdot 0 = 0 \cdot e.$$

Thus $\lambda_1 = 0$.

2 Graph Laplacians and Connectivity

Now we switch our focus to λ_2, which is much more interesting. We will see a very close connection between λ_2 and various notions of the connectivity of the graph.

Theorem 2 $\lambda_2 = 0$ *iff* G *is disconnected.*

Proof: If G is disconnected then, we can partition it into G_1 and G_2 such that there are no edges between G_1 and G_2. Furthermore, we can re-index the nodes so that

$$L_G = \begin{bmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{bmatrix}.$$

Then both vectors

$$\begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix}.$$

(where first $|V_{G_1}|$ entries of the first vector is 1 and the rest are zero and the opposite for the second vector) will be eigenvectors of L_G and orthogonal to each other. Since the eigenvalues associated with both vectors are 0, this implies that $\lambda_2 = 0$.

To see the other direction, let x_2 be an eigenvector of eigenvalue λ_2. We can assume $\langle x_2, e\rangle = 0$. If $\lambda_2 = 0$, then $x_2^T G x_2 = x_2^T(\lambda_2 x_2) = 0$. So then,

$$x_2^T L_G x_2 = \sum_{(i,j)\in E} (x_2(i) - x_2(j))^2 = 0.$$

The summation of squared real values is 0, therefore each of them is equal to zero. Therefore, $x_2(i) = x_2(j)$ for all $(i, j) \in E$. Consider $V_1 = \{i \in V : x_2(i) \geq 0\}$ and $V_2 = \{i \in V : x_2(i) < 0\}$. It's clear there are no edges between V_1 and V_2. Since $\langle x_2, e\rangle = 0$, there should be both positive and negative entries in x_2 which proves that $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, and hence G has at least two components. $\square$

For this reason λ_2 is called the *algebraic connectivity* of G. The proof above easily extends to prove the followng.

Claim 3 $\lambda_k = 0$ *iff* G *has at least* k *components.*

We now show another connection between λ_2 and the connectivity of the graph G.

Definition 2 $\kappa(G)$ *is the* vertex connectivity *of* G*; it is the smallest nonnegative integer such that we can remove up to* $\kappa(G) - 1$ *vertices and associated edges from* G *and* G *is still connected.*

We will show the following shortly. Let $G - S$ be the graph that results from removing the vertices in S from the graph, as well as all edges incident on the vertices in S.

Lemma 4 *$\lambda_2(L_G) \leq \lambda_2(L_{G-S}) + |S|$, for all $S \subseteq V$.*

Note that we easily get the following corollary.

Corollary 5 *$\lambda_2(G) \leq \kappa(G)$.*

Proof: Let S be a set of vertices of size $\kappa(G)$ that disconnects G. Then

$$\lambda_2(G - S) = 0 \Rightarrow \lambda_2(G) \leq 0 + \kappa(G).$$

$\square$

Proof of Lemma 4: Let x_2 be the eigenvector of L_{G-S} corresponding to $\lambda_2(L_{G-S})$, with $x_2^T x = 1$, $\langle x_2, e\rangle = 0$.

Then we know

$$x_2^T L_G x_2 = \sum_{(i,j)\in E} (x_2(i) - x_2(j))^2 = \lambda_2(L_{G-S})$$

for $G - S = (V - S, E')$. Note that $x_2 \in \Re^{|V|-|S|}$. We want a vector $x \in \Re^{|V|}$, so we let

$$x(i) = \begin{cases} x_2(i), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}.$$

With this definition x is a unit vector since, $x^T x = x_2^T x_2 = 1$ and $\langle x, e\rangle = \langle x_2, e\rangle = 0$. Then we have that

$$\begin{aligned}
\lambda_2(L_G) = \min_{z\in\Re^n:\langle z,e\rangle=0} \frac{z^T L_G z}{z^T z} &\leq \frac{x^T L_G x}{x^T x} \\
&= x^T L_G x \\
&= \sum_{(i,j)\in E} (x(i) - x(j))^2 \\
&= \sum_{(i,j)\in E'} (x(i) - x(j))^2 + \sum_{i\in S} \sum_{j:(i,j)\in E} (x(i) - x(j))^2 \\
&= \sum_{(i,j)\in E'} (x_2(i) - x_2(j))^2 + \sum_{i\in S} \sum_{j:(i,j)\in E} (x_2(j))^2 \\
&\leq \sum_{(i,j)\in E'} (x_2(i) - x_2(j))^2 + \sum_{i\in S} 1 \ (x_2 \text{ has unit norm}) \\
&= \lambda_2(L_{G-S}) + |S|.
\end{aligned}$$

$\square$

3 Graph Laplacians and Cuts

We now see that we can get some easy bounds on various types of cuts in graphs by using the eigenvalues of the Laplacian.

Definition 3 *If $|V|$ is even, let $b(G)$ be the smallest bisection of G; that is*

$$b(G) = min_{S \subset V : |S| = |V - S|} |\delta(S)|,$$

where $\delta(S)$ is the set of edges with one endpoint in S and the other endpoint in $V - S$. We say the edges in $\delta(S)$ are the edges in the cut *defined by S.*

Claim 6

$$\frac{n}{4}\lambda_2(G) \leq b(G).$$

Proof: Let $\bar{S}$ be an optimal bisection. Let $x\ in \{-1, +1\}^n$ s.t.

$$x(i) = \begin{cases} -1, & \text{if } i \in \bar{S} \\ +1, & \text{otherwise} \end{cases}.$$

Recall that

$$\lambda_2 = min_{z \in \Re^n, \langle z, e \rangle = 0} \frac{z^T L_G z}{z^T z}.$$

Note that $\langle x, e \rangle = 0$ since half of the entries of x are -1 and half are $+1$. Therefore,

$$\lambda_2 \leq \frac{x^T L_G x}{x^T x} = \sum_{(i,j) \in E} \frac{(x(i) - x(j))^2}{n} = \frac{1}{n} \cdot 4|\delta(\bar{S})| = \frac{4}{n} b(G).$$

□

To conclude the lecture, we turn to the largest eigenvalue of the Laplacian, and show that it has a connection to large cuts in the graph.

Definition 4 *Let $mc(G)$ be the maximum cut in the graph, so that*

$$mc(G) = max_{S \subseteq V} |\delta(S)|.$$

Then using the same idea as the proof above, we can show the following.

Claim 7

$$mc(G) \leq \frac{n}{4}\lambda_n(L_G).$$

Proof: Let $\bar{S}$ be a maximum cut and

$$x(i) = \begin{cases} -1, & \text{if } i \in \bar{S} \\ +1, & \text{otherwise} \end{cases}.$$

Then,

$$\lambda_n = \max_{z \in \Re^n} \frac{z^T L_G z}{z^T z} \geq \frac{x^T L_G x}{x^T x} = \sum_{(i,j) \in E} \frac{(x(i) - x(j))^2}{n} = \frac{4|\delta(\bar{S})|}{n} = \frac{4}{n} mc(G).$$

□

In fact, we can modify the bound above to give a tighter bound on the maximum cut.

Claim 8

$$mc(G) \leq \frac{n}{4} \min_{u : \langle u, e \rangle = 0} \lambda_n(L_G + diag(u)),$$

where $diag(u)$ is a diagonal matrix that $diag(u)(i,i) = u(i)$.

Proof: Following the same definition of x as above, we get that

$$\begin{aligned} \lambda_n(L_G + diag(u)) &= \max_{z \in \Re^n} \frac{z^T (L_G + diag(u)) z}{z^T z} \\ &\geq \frac{x^T L_G x + x^T diag(u)) x}{x^T x} \\ &= \frac{4mc(G) + \sum_{i \in V} u(i) x(i)^2}{n} \\ &= \frac{4mc(G)}{n}, \end{aligned}$$

since $x^2(i) = 1$ for all $i \in V$, and $\sum_{i \in V} u(i) = \langle u, e \rangle = 0$. □

This bound on the eigenvalue has a connection to other well-known bounds on the maximum cut problem. For a given vector u such that $\langle u, e \rangle = 0$, let $\lambda = \lambda_n(L_G + u)$. Define $\gamma(i) = \lambda - (u(i) + d(i))$ for all $i \in V$, where $d(i)$ is the degree of i in G. Then for adjacency matrix A, we have that

$$A + diag(\gamma) = \lambda I - (L_G + u).$$

Then we can see that $A + diag(\gamma) \succeq 0$ since for any $x \in \Re^n$,

$$\begin{aligned} x^T (A + diag(\gamma)) x &= x^T (\lambda I - (L_G + u)) x \\ &= \lambda x^T x - x^T (L_G + u) x \\ &\geq x^T (L_G + u) x - x^T (L_G + u) x \\ &= 0, \end{aligned}$$

where the inequality follows since $\lambda \geq x^T(L_G + u)x/x^Tx$. Then we observe that

$$\begin{aligned}\frac{n}{4}\lambda &= \frac{1}{4}\sum_{i \in V}(\gamma(i) + u(i) + d(i)) \\ &= \frac{1}{4}\sum_{i \in V}\gamma(i) + \frac{1}{4}\sum_{i \in V} d(i) \\ &= \frac{1}{4}\sum_{i \in V}\gamma(i) + \frac{1}{2}|E|.\end{aligned}$$

Then finding a u to minimize $\frac{n}{4}\min_{u:\langle u,e\rangle=0}\lambda_n(L_G + diag(u))$ turns out to be equivalent to finding a γ to minimize

$$\frac{1}{4}\sum_{i \in V}\gamma(i) + \frac{1}{2}|E|,$$

subject to

$$A + diag(\gamma) \succeq 0.$$

This is a *semidefinite program*, and it has a dual semidefinite program of maximizing

$$\frac{1}{2}\sum_{(i,j) \in E}(1 - x_{ij})$$

subject to

$$x_{ii} = 1 \text{ for all } i \in V, \qquad X = (x_{ij}) \succeq 0.$$

This semidefinite program is used in a .878-approximation algorithm for the maximum cut problem due to Goemans and W. Thus one can show that the eigenvalue bound is a strong one; we also have that

$$mc(G) \geq .878 \cdot \frac{n}{4}\min_{u:\langle u,e\rangle=0}\lambda_n(L_G + diag(u)).$$