ORIE 6334 Spectral Graph Theory	September 1, 2016
Lecture 4	
Lecturer: David P. Williamson	*Scribe: Kun Dong*

Once again, we'll start by proving a general theorem about eigenvalues, and then show its application to some graph problems.

1 Eigenvalue Interlacing Theorem

The following theorem is known as the *eigenvalue interlacing theorem.*

Theorem 1 (Eigenvalue Interlacing Theorem) *Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric. Let $B \in \mathbb{R}^{m\times m}$ with $m < n$ be a principal submatrix (obtained by deleting both i-th row and i-th column for some values of i). Suppose A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and B has eigenvalues $\beta_1 \leq \cdots \leq \beta_m$. Then*

$$\lambda_k \leq \beta_k \leq \lambda_{k+n-m} \quad \textit{for} \quad k = 1, \cdots, m$$

And if $m = n - 1$,

$$\lambda_1 \leq \beta_1 \leq \lambda_2 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \lambda_n$$

Proof: WLOG, assume $A = \begin{bmatrix} B & X^T \\ X & Z \end{bmatrix}$. Let $\{x_1, \cdots, x_n\}$ be eigenvectors of A, $\{y_1, \cdots, y_m\}$ be eigenvectors of B. We define the following vector spaces:

$$V = span(x_k, \cdots, x_n), \qquad W = span(y_1 \cdots, y_k), \qquad \widetilde{W} = \left\{ \begin{pmatrix} w \\ 0 \end{pmatrix} \in \mathbb{R}^n, w \in W \right\}$$

Since $\dim(V) = n - k + 1$ and $\dim(\widetilde{W}) = \dim(W) = k$, there exists $\widetilde{w} \in V \bigcap \widetilde{W}$ and $\widetilde{w} = \begin{pmatrix} w \\ 0 \end{pmatrix}$ for some $w \in W$. Then

$$\widetilde{w}^T A \widetilde{w} = \begin{bmatrix} w^T & 0 \end{bmatrix} \begin{bmatrix} B & X^T \\ X & Z \end{bmatrix} \begin{bmatrix} w \\ 0 \end{bmatrix} = w^T B w$$

Recall $\lambda_k = \min_{x\in V} \dfrac{x^T A x}{x^T x}$ and $\beta_k = \max_{x\in W} \dfrac{x^T B x}{x^T x}$. Then we see that

$$\lambda_k \leq \frac{\widetilde{w}^T A \widetilde{w}}{\widetilde{w}^T \widetilde{w}} = \frac{w^T B w}{w^T w} \leq \beta_k$$

[0]This lecture was drawn from some notes of Embree `http://www.caam.rice.edu/~caam440/chapter2.pdf` and Spielman's 2012 lecture notes, Lecture 3: `http://www.cs.yale.edu/homes/spielman/561/2012/lect03-12.pdf`.

The proof of the other inequality is similar. We now define the vector spaces

$$V = span(x_1, \cdots, x_{k+n-m}), \qquad W = span(y_k \cdots, y_m), \qquad \widetilde{W} = \left\{ \begin{pmatrix} w \\ 0 \end{pmatrix} \in \mathbb{R}^n, w \in W \right\}$$

Since $\dim(V) = k + n - m$, $\dim(\widetilde{W}) = \dim(W) = m - k + 1$, there exists $\widetilde{w} \in V \bigcap W$ and $\widetilde{w} = \begin{pmatrix} w \\ 0 \end{pmatrix}$ for some $w \in W$. We have $\widetilde{w}^T A \widetilde{w} = w^T B w$ as before. It follows that

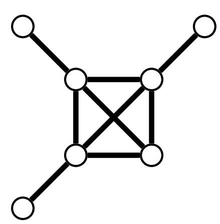
$$\lambda_{k+n-m} = \max_{x \in V} \frac{x^T A x}{x^T x} \geq \frac{\widetilde{w}^T A \widetilde{w}}{\widetilde{w}^T \widetilde{w}} = \frac{w^T B w}{w^T w} \geq \min_{x \in W} \frac{x^T B x}{x^T x} = \beta_k,$$

completing the proof. □

2 Clique and Chromatic Number

We now use the eigenvalue interlacing theorem to prove some statements about two particular graph quantities, the clique number and the chromatic number.

Definition 1 *The clique number of G, $\omega(G)$, is the size of the largest $S \subseteq V$ such that for all $i, j \in S$, $(i, j) \in E$.*
Example:



$\omega(G) = 4$

Definition 2 *The chromatic number $\chi(G)$ is the fewest number of colors needed such that we can assign one color to each vertoex and for all $(i, j) \in E$, i, j are assigned different colors.*

Observation 1 *$\chi(G) \geq \omega(G)$.*

The observation follows since every vertex in the maximum clique needs to be assigned a different color: if two vertices in the clique are assigned the same color, then since there is an edge between them, the two endpoints of that edge are not assigned different colors.

Consider the complete graph on n nodes $G \equiv K_n$; that is, there is an edge between every pair of vertices. Then $\omega(G) = n = \chi(G)$. The adjacency matrix of G is $A = J - I$ where J is the matrix of all ones. Let

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then

$$Ae = (J - I)e = ne - e = (n-1)e$$

Therefore e is an eigenvector for eigenvalue $n-1$.

For any vector v such that $\langle e, v \rangle = 0$, $Av = (J - I)v = 0 - v = -v$. This means any v such that $\langle e, v \rangle = 0$ is an eigenvector of eigenvalue -1. Thus the spectrum of A is $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$.

Now consider an arbitrary graph G, and let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues to its adjacency matrix.

Claim 2 $\lambda_1 \geq \omega(G) - 1$.

Proof: For the largest clique S in G, let B be the principal submatrix with columns and rows corresponding to S. Let $m = |S| = \omega(G)$, then $B = J_m - I_m$. If β_1 is the largest eigenvalue of B, then $\beta_1 = \omega(G) - 1$. By the Interlacing Theorem, $\lambda_1 \geq \beta_1 = \omega(G) - 1$. □

We can in fact prove something slightly stronger. The following theorem strengthens that bound of the claim since $\omega(G) \leq \chi(G)$.

Theorem 3 (Wilf 1967) $\chi(G) \leq \lfloor \lambda_1 \rfloor + 1$

Before we can prove this we need a lemma. Let $d(i)$ be the degree of node i, $\Delta = \max_i d(i)$ and $d_{ave} = \frac{\sum_{i \in V} d(i)}{n}$.

Lemma 4 $d_{ave} \leq \lambda_1 \leq \Delta$ *for G connected.*

Proof:

$$\lambda_1 = \max_x \frac{x^T A x}{x^T x} \geq \frac{e^T A e}{e^T e} = \frac{\sum_{i,j} a_{ij}}{n} = \frac{\sum_{i \in V} d(i)}{n} = d_{ave}$$

Let x_1 be the eigenvector. By Perron-Frobenius Theorem, we can assume $x_1 > 0$. Let $i^* = arg \max_i x(i)$, then

$$(Ax_1)(i^*) = \lambda_1 x_1(i^*)$$

$$\Longrightarrow \lambda_1 = \frac{(Ax_1)(i^*)}{x_1(i^*)} = \frac{\sum_{j:(j,i^*) \in E} x_1(j)}{x_1(i^*)} = \sum_{j:(j,i^*) \in E} \frac{x_1(j)}{x_1(i^*)} \leq \sum_{j:(j,i^*) \in E} 1 = d(i^*) \leq \Delta$$

□

Notice that we needed connectivity in the proof above to invoke the Perron-Frobenius theorem for the inequality $\lambda_1 \leq \Delta$; we did not need it for the lower bound $d_{ave} \leq \lambda_1$.

As an aside, we can use the above lemma to prove the following, though we will not do so here.

Claim 5 *G is connected and $\lambda_1 = \Delta$, then G is Δ-regular.*

We also observe the following.

Observation 2 $\chi(G) \leq \Delta + 1$.

This is true because if we color the graph greedily, we will never get stuck: if we color a vertex, it has at most Δ neighbors that have already been colored, and so we can color it with the $(\Delta + 1)$st color.

Proof of Theorem 3:

The proof is by induction on n.

Base case $n = 2$,

 $\lambda_1 = 1, \quad \chi_1(G) = 2$

 $\lambda_1 = 0, \quad \chi_1(G) = 1$

Inductive step: Suppose the theorem holds on all graphs with at most $n - 1$ vertices. By the Lemma, G has a vertex of degree less than $\lfloor \lambda_1 \rfloor$. Remove this vertex v and call the resulting graph G'. Let B be its adjacency matrix and β_1 be its largest eigenvalue. By the Interlacing Theorem, $\beta_1 \leq \lambda_1$. By induction, we can color G' with $\lfloor \beta_1 \rfloor + 1$ colors, which is less than $\lfloor \lambda_1 \rfloor + 1$ colors. We can then finish coloring G by coloring v with one of the $\lfloor \lambda_1 \rfloor + 1$ colors since degree of v is less than $\lfloor \lambda_1 \rfloor$. $\square$

We now give one more result, a lower bound on the clique number. In order to prove it, we assume the following theorem of Motzkin and Straus.

Theorem 6 (Motzkin-Straus 1965)

$$\begin{aligned} \frac{1}{2}\left(1 - \frac{1}{\omega(G)}\right) = \quad & \textit{Maximize} \quad \sum_{(i,j)\in E} x_i x_j \\ & \textit{subject to:} \\ & \qquad \sum_{i \in V} x_i \leq 1 \\ & \qquad x_i \geq 0 \qquad\qquad \forall i \in V. \end{aligned}$$

Then we can prove the following

Theorem 7 (Wilf 1986) *Let $S = \sum_{i=1}^{n} x_1(i)$ where x_1 is the eigenvector for the largest eigenvalue and $\|x_1\|^2 = 1$. Then $\omega(G) \geq \frac{S^2}{S^2 - \lambda_1}$.*

Proof: Let $\overline{x}$ be solution to Motzkin-Straus program. Then

$$1 - \frac{1}{\omega(G)} = \overline{x}^T A \overline{x};$$

notice that the factor of $\frac{1}{2}$ is dropped because $\overline{x}^T A \overline{x}$ counts every edge twice. By the Perron-Frobenius Theorem, we can assume $x_1 \geq 0$ and $\frac{x_1}{S}$ is feasible for the program. This implies

$$1 - \frac{1}{\omega(G)} = \overline{x}^T A \overline{x} \geq \frac{x_1}{S}^T A \frac{x_1}{S} = \frac{\lambda_1}{S^2}$$

so that

$$1 - \frac{\lambda_1}{S^2} \geq \frac{1}{\omega(G)},$$

which implies

$$\frac{S^2 - \lambda_1}{S^2} \geq \frac{1}{\omega(G)},$$

or

$$\omega(G) \geq \frac{S^2}{S^2 - \lambda_1}.$$

$\square$

Using the Cauchy-Schwartz inequality, we can get that

$$S = \sum_{i=1}^{n} x_1(i) \leq \|x_1\| \sqrt{1^2 + \cdots + 1^2} = \sqrt{n}.$$

Plugging this bound into the inequality above gives us

$$\omega(G) \geq \frac{n}{n - \lambda_1}.$$