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Lecture 3

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1 Eigenvalue Identities

We first present a few useful eigenvalue identities. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with real eigenvalues $\lambda_1 \geq \lambda_2 \geq ...\lambda_n$ with corresponding eigenvectors $x_1, x_2, \ldots, x_n$ such that the x_i are orthonormal.

Lemma 1 *The eigenvectors of A^k are $x_1, \ldots, x_n$ with corresponding eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$.*

Proof: Let

$$X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

Then,

$$AX = XD$$

since the columns of X are the eigenvectors. Interestingly, $X^TX = I$ as the columns are orthonormal, so $X^T = X^{-1}$. This implies $A = XDX^{-1}$ (right multiplying $AX = DX$ by X^{-1}). Then, $A^k = (XDX^{-1})^k = XD^kX^{-1}$, where D^k has the form

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^k \end{bmatrix}$$

It follows that $A^kX = XD^k$; therefore, the eigenvectors of A^k are $x_1, \ldots, x_n$ with corresponding eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$. $\square$

For our next few identities, we need the following fact that we present without proof.

Fact 1 $\det(AB) = \det(A)\det(B)$

An easy corollary of this fact is $\det(A^{-1}) = \frac{1}{\det(A)}$ since:

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

[0]This lecture was drawn from Lau's 2015 lecture notes, Lecture 1:`https://cs.uwaterloo.ca/~lapchi/cs798/notes/L01.pdf` and Spielman's 2012 lecture notes, Lecture 3: `http://www.cs.yale.edu/homes/spielman/561/2012/lect03-12.pdf`.

Lemma 2

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

Proof:

$$\begin{aligned}\det(A) &= \det(XDX^{-1}) \\ &= \det(X)\det(D)\det(X^{-1}) \\ &= \det(D) \\ &= \prod_{i=1}^{n} \lambda_i.\end{aligned}$$

□

Recall that in the first lecture, we defined the *trace* of A to be $Tr(A) = \sum_{i=1}^{n} a_{ii}$. We used the following without proof in the first lecture, and now we can prove it.

Lemma 3

$$Tr(A) = \sum_{i=1}^{n} \lambda_i.$$

Proof: Consider the characteristic polynomial of A, which we defined in the first lecture to be $\det(\lambda I - A)$; it is a degree n polynomial in λ. Then we can rewrite it as:

$$\begin{aligned}\det(\lambda I - A) &= \det(\lambda X X^T - XDX^T) \\ &= \det(X(\lambda I - D)X^T) \\ &= \det(X)\det(\lambda I - D)\det(X^T) \\ &= \det(\lambda I - D) \\ &= \prod_{i=1}^{n}(\lambda - \lambda_i).\end{aligned}$$

We see that indeed the eigenvalues are precisely the roots of the polynomial. Here, the coefficient of λ^{n-1} is exactly $-\sum_{i=1}^{n} \lambda_i$.

Now, recall that the determinant of a matrix Z with permutations S_n is:

$$\det(Z) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

The only permutation that can produce a λ^{n-1} term is the identity, so the only term in the sum with a λ^{n-1} term is $\prod_{i=1}^{n}(\lambda - a_{ii})$. The coefficient of λ^{n-1} here is $-\sum_{i=1}^{n} a_{ii}$. Then, we are done, with

$$Tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i.$$

□

2 The Perron-Frobenius Theorem

Recall for an undirected graph G, its *adjacency matrix* is defined as $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Theorem 4 (Perron-Frobenius) *Let G be a connected graph with adjacency matrix A, eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and corresponding eigenvectors $x_1, \ldots, x_n$. Then:*

(i) $\lambda_1 \geq -\lambda_n$;

(ii) $\lambda_1 > \lambda_2$;

(iii) There exists an eigenvector $x_1 > 0$ (that is, every coordinate is strictly positive).

Proof: First we will prove (iii). Let $x_1, \ldots, x_n$ be the corresponding eigenvectors, and assume they are orthonormal. Recall that

$$\lambda_1 = \max_{x \in \mathbb{R}^n} \frac{x^T A x}{x^T x} = x_1^T A x_1.$$

Define a vector y such that $\forall i, y(i) = |x_1(i)|$; then $y^T y = x_1^T x_1 = 1$. We show that y is also an eigenvector corresponding to λ_1. To see this, we have

$$\begin{aligned}
\lambda_1 &= x_1^T A x_1 \\
&= \sum_{ij} a_{ij} x_1(i) x_1(j) \\
&\leq \sum_{ij} a_{ij} |x_1(i)||x_1(j)| \\
&= \sum_{ij} a_{ij} y(i) y(j) \\
&= y^T A y \\
&\leq \lambda_1.
\end{aligned}$$

The last inequality follows by the definition of λ_1, and by the fact that y has unit norm. Since $\lambda_1 = \lambda_1$, all the inequalities must be equalities, so y is an eigenvector of λ_1.

We now argue that none of the entries of y can be zero. We have $y \geq 0$ by definition and $y \neq 0$ since it is an eigenvector. To show that none of the entires are zero, we use the fact that the graph is connected, so if there is some j such that $y(j) = 0$, this means that there is an edge $(i,k) \in E$ such that $y(i) = 0$ and $y(k) \neq 0$. Then we have:

$$(Ay)(i) = \sum_{j:(i,j) \in E} y(j) \geq y(k) > 0$$

However, $(Ay)(i) = \lambda_i y(i) = 0$, which is a contradiction.

Now we prove (i). Let $\forall i$, $y(i) = |x_n(i)|$. Again, we have

$$y^T y = x_n^T x_n = 1$$

so the vector y has unit norm. Then

$$\begin{aligned}
|\lambda_n| &= |x_n^T A x_n| \\
&\leq \sum_{ij} a_{ij} |x_n(i)||x_n(j)| \\
&= \sum_{ij} a_{ij} y(i) y(j) \\
&= y^T A y \\
&\leq \lambda_1,
\end{aligned}$$

as desired.

For (ii), let $\forall i$, $y(i) = |x_2(i)|$. Then $y^y = x_2^T x_2 = 1$. Then we have that

$$\begin{aligned}
\lambda_2 &= x_2^T A x_2 \\
&\leq \sum_{ij} a_{ij} |x_2(i)||x_2(j)| \\
&= \sum_{ij} a_{ij} y(i) y(j) \\
&= y^T A y \\
&\leq \lambda_1,
\end{aligned}$$

Now we show that somewhere along the way, the inequality is strict. Assume $x_1 > 0$, as we can from (iii). Since $\langle x_1, x_2 \rangle = 0$, and both are nonzero, some of the entries of x_2 are positive and some are negative. We split into two cases:

- **Case 1:** All of the entries of x_2 are nonzero. Then, since G is connected, $\exists (i,j) \in E$ such that $x_2(i) < 0, x_2(j) > 0$. Then, $x_2(i)x_2(j) < |x_2(i)||x_2(j)|$ which gives us the strict inequality that we wanted. Hence, $\lambda_2 < \lambda_1$.

- **Case 2:** $x_2(i) = 0$ for some i. If all inequalities are equalities, y is an eigenvector of λ_1 with $y \geq 0$. We argued above that when $y \geq 0$ and is an eigenvector corresponding to λ_1 and G is connected, then none of the entries of y can be zero. But since $y(i) = x_2(i) = 0$ for some i, this is a contradiction.

$\square$

3 Bipartite Graphs

We now turn to showing how all the various identities we've proven over this lecture and the last can be applied to showing something about the structure of graphs. In particular, we show that the spectrum of the adjacency matrix tells us whether the graph is bipartite or not.

Lemma 5 *If G is bipartite, and λ is an eigenvalue of adjacency matrix A, then so is $-\lambda$.*

Proof: If G is bipartite, we can re-index the nodes such that

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

Then let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvector of A with eigenvalue λ. Then, we have

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence we have $By = \lambda x$ and $B^T x = \lambda y$ So from this,

$$\begin{aligned} \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} &= \begin{bmatrix} -By \\ B^T x \end{bmatrix} \\ &= \begin{bmatrix} -\lambda x \\ \lambda y \end{bmatrix} \\ &= -\lambda \begin{bmatrix} x \\ -y \end{bmatrix} \end{aligned}$$

So, $-\lambda$ is an eigenvalue corresponding to the eigenvector $[x - y]^T$. □

We can now show that this is statement can be made an "if and only if": that is the graph G is bipartite if and only if for each eigenvalue λ there is another eigenvalue $-\lambda$.

Theorem 6 *If for each eigenvalue $\lambda \neq 0$ there is another eigenvalue $\lambda' = -\lambda$, then G is bipartite.*

Proof: Let k be any odd positive integer. Then by hypothesis,

$$Tr(A^k) = \sum_{i=1}^{n} \lambda_i^k = 0$$

It can be shown by induction that $(A^k)_{ij}$ is the number of walks from i to j of length exactly k (recall from the first lecture that we used that $(A^2)_{ij} = \sum_k a_{ik} a_{kj}$ is the number of walks of length exactly two, using an edge from i to k then k to j). Notice that if there is an odd cycle of length k, then it must be the case that $(A^k)_{ii} > 0$, so that $Tr(A^k) > 0$. But since $Tr(A^k) = 0$, there are no odd cycles of length k. Since this is true for any odd positive integer k, there are no odd cycles in G, which implies that G is bipartite. □

Now we can show something even stronger than the previous statement: we only need to look at the smallest and largest eigenvalue to know whether or not the graph is bipartite.

Theorem 7 *Suppose G is connected. Then, $\lambda_n = -\lambda_1$ if and only if G is bipartite.*

Proof: By Perron-Frobenius, $\lambda_1 \geq -\lambda_n$, and by the previous theorem, the graph being bipartite implies that $\lambda_1 = -\lambda_n$.

For the other direction, Let x_n be the eigenvector corresponding to λ_n with $x_n^T x_n$. Let $y(i) = |x_n(i)|$ for all i. Again, we have $y^T y = x_n^T x_n = 1$. Also,

$$\begin{aligned}
|\lambda_n| &= |x_n^T A x_n| \\
&\leq \sum_{ij} a_{ij} |x_n(i)||x_n(j)| \\
&= \sum_{ij} a_{ij} y_n(i) y_n(j) \\
&= y^T A y \\
&\leq \lambda_1.
\end{aligned}$$

The assumption $\lambda_n = -\lambda_1$ implies that all the inequalities are equalities. This implies that y is an eigenvector corresponding to λ_1, with $y \geq 0$. By our proof of Perron-Frobenius, since $y \geq 0$, we have $y > 0$ and this implies that $x_n(i) \neq 0$ for all i.

If all the inequalities are equalities, $x_n(i)x_n(j)$ has the same sign whenever $a_{ij} > 0$. Since $\lambda_n = x_n^T A x_n < 0$, all of these products must be negative. This implies that for any edge in the graph, either $x_n(i) > 0, x_n(j) < 0$ or $x_n(i) < 0, x_n(j) > 0$. This induces the bipartition

$$\begin{aligned}
V &= \{i : x_n(i) < 0\}, \\
W &= \{i : x_n(i) < 0\}.
\end{aligned}$$

□

This brings about an interesting research question. What happens when λ_1 is close to $-\lambda_n$? Does this mean that the graph is "almost bipartite", in the sense that if we remove some of the edges, it would become bipartite? Possibly the answer to this question is already well known.