

This is an attempt to recap what we have discovered thus far. We have been considering the problem of finding a sparse cut in an undirected graph with capacities u_e , subject to all-pairs demands. That is, we would like to find

$$\alpha = \min_{S \subset V} \frac{\sum_{e \in \delta(S)} u_e}{|S||V-S|}.$$

As was observed by Michel, this problem can be relaxed to the following problem that can be solved in polynomial time:

$$\begin{aligned} \alpha_N^* = & \text{Min} \sum_{e \in E} u_e x_e \\ \text{subject to:} & \\ (NEG) \quad & \sum_{i \neq j} d_x(i, j) \geq 1 \\ & d_x \text{ of negative type,} \end{aligned}$$

where d_x is a distance metric under edge lengths x . A negative type metric obeys the inequalities $\sum_{i,j} d_{ij} z_i z_j \leq 0$ for all vectors z such that $\sum_i z_i = 0$. We note that this is a relaxation of the sparsest cut problem since for an optimal cut S , we can set $x_e = 1/|S||V-S|$ if $e \in \delta(S)$ and $x_e = 0$ otherwise. Then the objective function has the right value, and $\sum_{i \neq j} d_x(i, j) \geq 1$. Furthermore, for any z such that $\sum_i z_i = 0$,

$$\sum_{i,j} d_{ij} z_i z_j = \frac{1}{|S||V-S|} \sum_{i \in S, j \notin S} z_i z_j = \frac{1}{|S||V-S|} \left(\sum_{i \in S} z_i \right) \left(- \sum_{i \in S} z_i \right) \leq 0.$$

We note in passing that the relaxation used by Leighton and Rao (and hence by Linial, London, and Rabinovich) is the same as the one above without the negative type restriction. Let α_{LR}^* denote the value of their relaxation. Certainly $\alpha_{LR}^* \leq \alpha_N^*$; we will show below that there are important instances for which $\alpha_N^* > \alpha_{LR}^*$.

We give our main result (thus far) below. For any non-negative matrix M with $m_{ii} = 0$, we denote the Laplacian of the matrix $L(M)$ as

$$L(M)_{ij} = \begin{cases} \sum_k m_{ik} & i = j \\ -m_{ij} & i \neq j \end{cases}$$

Theorem 1 *Let f be any flow in the graph G obeying capacities u but not necessarily flow conservation. Let F be a matrix such that F_{ij} denotes the amount of flow from i to j . Then if λ_2 is the second smallest eigenvalue of $L(F)$, $\alpha_N^* \geq \lambda_2/n$, and there exists a flow F such that $\alpha_N^* = \lambda_2/n$.*

A consequence of this theorem is that bounded-degree expanders (not necessary regular ones!) with constant expansion ratio are not a bad case for the negative type relaxation. Leighton and Rao have shown that $\alpha > \Omega(\log n) \alpha_{LR}^*$ for bounded-degree expanders. However, consider the flow on a bounded-degree expander in which each vertex sends one unit of flow to each of its neighbors. This is a feasible flow in this graph [Modulo having edges of the appropriate capacity in each direction...] Then the Laplacian of this flow matrix is simply $L(A)$, where A is the adjacency matrix of the expander. If γ is the expansion ratio of the graph, then the sparsest cut has value roughly $O(\gamma/n)$, and λ_2 of the Laplacian is known to have value $\Omega(\gamma^2)$ (Alon).

Proof: As one might expect from the theorem statement, the theorem follows by duality. A more detailed description of the relaxation (NEG) is the following:

$$\begin{aligned} \alpha_N^* = & \text{Min} \sum_{e \in E} u_e x_e \\ \text{subject to:} & \\ (NEG') \quad & \sum_{i \neq j} d_{ij} \geq 1 \end{aligned}$$

$$\begin{aligned}
\sum_{e \in P_{ab}^k} x_e &\geq d_{ab} & \forall a, b \text{-paths } P_{ab}^k \\
\sum_{i,j} d_{ij} z_i z_j &\leq 0 & \forall z, \sum_i z_i = 0, \\
d_{ij} &\geq 0 & \forall i \neq j \\
x_e &\geq 0 & \forall e \in E
\end{aligned}$$

where P_{ab}^k is the k th path from a to b in an enumeration of all a, b paths. Taking the dual, we obtain

$$\begin{aligned}
\alpha_N^* &= \text{Max } w \\
\text{subject to:} \\
(D) \quad w - \sum_k f_{ij}^k - \sum_z a_z z_i z_j &\leq 0 & \forall i \neq j \\
\sum_{k,a,b:e \in P_{ab}^k} f_{ab}^k &\leq u_e & \forall e \in E \\
a_z &\geq 0 & \forall z, \sum_i z_i = 0 \\
f_{ij}^k &\geq 0 & \forall i, j, k \\
w &\geq 0
\end{aligned}$$

Observe that f_{ij}^k corresponds to a flow on the k th path from i to j , such that the flows obey the capacity constraints. We set $f_{ij} = \sum_k f_{ij}^k$, and $F = (f_{ij})$. Notice that since a_z is positive and non-zero only when $z^T \mathbf{1} = 0$, then $\sum_z a_z z^T z = M$, where M is a positive semidefinite matrix such that $M\mathbf{1} = 0$. Since the a_z do not participate in the objective function, there are no constraints on the diagonal, and adding any sufficiently large diagonal to any matrix makes it positive semidefinite, we can rewrite the relaxation compactly in matrix notation as follows:

$$\begin{aligned}
\alpha_N^* &= \text{Max } w \\
\text{subject to:} \\
(D') \quad w(J - I) - F + \text{diag}(u_i) &\text{ psd} \\
[w(J - I) - F + \text{diag}(u_i)]^T \mathbf{1} &= 0 \\
F &\text{ a flow obeying capacity constraints} \\
w &\geq 0,
\end{aligned}$$

where J is the all 1's matrix, and I is the identity. To prove our theorem, let F be any feasible flow; we will try to find the maximum value x allowed by such a flow. We set $u_i = \sum_j f_{ij} - x(n-1)$ so that $x(J - I) - F + \text{diag}(u_i) = xJ - xnI + L(F)$. Then $[xJ - xnI + L(F)]^T \mathbf{1} = 0$. Notice that $L(F)$ has an eigenvalue of 0 for the eigenvector $\mathbf{1}$. Let v_2 be the eigenvector for the second smallest eigenvalue of $L(F)$, λ_2 . Then since $v_2^T \mathbf{1} = 0$, v_2 is also the eigenvector of the second smallest eigenvalue of $xJ - xnI + L(F)$, of value $-xn + \lambda_2$. So the matrix $xJ - xnI + L(F)$ is psd iff $x \leq \lambda_2/n$. Thus we can set $x = \lambda_2/n$, and the theorem follows. ■

We can say something about the optimality of the relaxation in one particular case (so far).

Theorem 2 *Let F be a path-packing that attains the maximum of $\max_F \lambda_2(L(F))/n$. If $\lambda_2(L(F)) < \lambda_3(L(F))$, then $\lambda_2(L(F))/n = \alpha$.*

Proof: An alternate formulation of $\lambda_2(L(F))/n$ (see Mohar and Poljak) is

$$\min_{x \neq c\mathbf{1}} \frac{\sum_{i,j} f_{ij}(x_i - x_j)^2}{\sum_{i,j} (x_i - x_j)^2}.$$

If F attains the maximum and the second eigenvalue of $L(F)$ is unique and achieved by a vector x , then we know that: (1) no perturbation of F can increase $\lambda_2(L(F))$; (2) no perturbation of x can decrease $\lambda_2(L(F))$.

Consider the ordering imposed by the x_i : assume that $x_1 \leq \dots \leq x_n$. We can show that F must have a highly specific form. By (1), for any f_{ij} , $i < j$, all paths packed between i and j must visit vertices in increasing order. Suppose not, and for some path from i to j , visits i_1 then i_2 , where $i_1 > i_2$. Then we can increase $\lambda_2(L(F))$ by splitting the path from i to j into 3 paths: the path from i to i_1 , the path from i_2 to i_1 , and the path from i_2 to j . By (1) we can also show that for any i , either $f_{ij} = 0$ for $j > i$ or $f_{ji} = 0$ for $j < i$: if there exists $a < i < b$ such that $f_{ai} > 0$ and $f_{ib} > 0$, then we can increase $\lambda_2(L(F))$ by taking ϵ of the paths from a to i and i to b and changing it into paths from a to b . Finally, the path-packings must be nested in the sense that it cannot be the case that $f_{ac} > 0$ and $f_{bd} > 0$ for $a < b < c < d$ and that $f_{ac} > 0$ and $f_{bc} > 0$ for $a < b < c$. Suppose that one of the two can happen: overall all such quadruplets and triplets, let $a < b < c < d$ be the quadruplet that minimizes $c - b$ or $a < b < c$ the triplet that minimizes $c - b$. Then we can decrease $\lambda_2(F)$ by swapping x_b and x_c .