ORIE 6334 Approximation Algorithms	November 12, 2009

Problem Set 5

Due Date: December 1, 2009

1. W&S Exercise 6.5
2. W&S Exercise 11.2
3. W&S Exercise 11.3
4. Consider the *generalized assignment problem*: we are given a set J of n jobs, and a set M of m machines. There is a cost $c_{ij} \geq 0$ for assigning job j to machine i; if job j is assigned to machine i, it requires $p_{ij} \geq 0$ units of processing time on machine i. We are also given as input processing bounds T_i for all machines $i \in M$. The goal is to find an assignment of jobs to machines that minimizes the total cost of the assignment, and such that the total amount of processing assigned to machine i is at most T_i for all $i \in M$.

   We wish to give a linear programming relaxation of the problem. We let E denote a set of possible (i,j) pairs that we could make in the assignment. Initially, E consists of all (i,j) such that $i \in M$, $j \in J$ and $p_{ij} \leq T_i$. We also have a subset $M' \subseteq M$, where initially $M' = M$, and a subset $J' \subseteq J$, where initially $J' = J$. Then the relaxation is as follows:

$$\begin{array}{lrcll} \text{Min} & \displaystyle\sum_{(i,j)\in E} c_{ij}x_{ij} & & & \\ & \displaystyle\sum_{i\in M:(i,j)\in E} x_{ij} & = & 1 & \forall j \in J' \\ & \displaystyle\sum_{j\in J:(i,j)\in E} p_{ij}x_{ij} & \leq & T_i & \forall i \in M' \\ & x_{ij} & \geq & 0 & \forall (i,j) \in E. \end{array}$$

   Consider the following algorithm: While $J' \neq \emptyset$, we find a basic solution to the LP relaxation. We remove from E any pair (i,j) such that $x_{ij} = 0$. If there is a variable $x_{ij} = 1$, then we assign job j to machine i; remove j from J' and reduce T_i by p_{ij}. Let J_i be the jobs fractionally assigned to machine $i \in M'$, so that $J_i = \{j \in J : x_{ij} > 0\}$. If there is a machine i such that $\sum_{j\in J_i} x_{ij} \geq 1$ and $1 \leq |J_i| \leq 2$, then remove i from M'.

   (a) Prove that for any basic solution x to the LP, either there is some $(i,j) \in E$ such that $x_{ij} \in \{0,1\}$, or there exists some $i \in M'$ with $\sum_{j\in J_i} x_{ij} \geq 1$ and $1 \leq |J_i| \leq 2$.

You may assume the following for any basic solution x to the linear program: there exist subsets $J'' \subseteq J'$ and $M'' \subseteq M'$ such that the LP constraint $\sum_{j \in J:(i,j)\in E} p_{ij}x_{ij} = T_i$ for all $i \in M''$, the vectors corresponding to the LP rows for J'' and M'' are linearly independent, and $|J''| + |M''|$ is equal to the number of variables $x_{ij} > 0$.

(b) Prove that the algorithm above returns a solution with total cost at most OPT, and such that machine i is assigned total processing time $2T_i$ for all $i \in M$.