ORIE 6334 Approximation Algorithms October 1, 2009

Problem Set 3

Due Date: October 24, 2009

1. W&S Exercise 5.2
2. W&S Exercise 5.3
3. W&S Exercise 5.5
4. Consider the problem of maximizing a non-negative submodular function $f : 2^V \rightarrow \Re^{\geq 0}$. Suppose we select a random set S by including each $i \in V$ with probability 1/2. We will show that $E[f(S)] \geq \frac{1}{4}\,\mathrm{OPT}$.

   (a) Given an $A \subseteq V$, let $A(p)$ be the random set of elements obtained by including each $i \in A$ with probability p. Prove that

   $$E[f(A(p))] \geq (1-p)f(\emptyset) + pf(A).$$

   (b) Now prove that for $A \subseteq V$ and $B \subseteq V$, prove that

   $$E[f(A(p)\cup B(q))] \geq (1-p)(1-q)f(\emptyset)+p(1-q)f(A)+(1-p)qf(B)+pqf(A\cup B).$$

   (Hint: for a fixed A', consider the function $g(T) = f(A' \cup T)$. Prove that g is submodular. What is $E[g(B(q))]$?)

   (c) Let $O \subseteq V$ be an optimal set so that $f(O) = \mathrm{OPT}$. Apply the previous part to the case $A = O$, $B = V - O$, and $p = q = 1/2$ to conclude that $E[V(1/2)] \geq \frac{1}{4}\,\mathrm{OPT}$.

   (d) What can you show if f is symmetric (i.e. $f(S) = f(V - S)$ for all $S \subseteq V$)?

5. W&S Exercise 6.2