Defn An arc (i,j) is <u>ε-fixed</u> if flow $f(i,j)$ is the same for all ε-optimal circulations f.

Lemma Let $\varepsilon > 0$, f be a circ, p be potentials s.t. f is ε-optimal wrt p. If $c_p(i,j) \leq -2n\varepsilon$, then (i,j) is ε-fixed.

Lemma Let f and f' be circulations s.t. $\varepsilon(f') \leq \varepsilon(f)/2n$ and f is not a min-cost circ. Then there are strictly more $\varepsilon(f')$-fixed arcs than $\varepsilon(f)$-fixed arcs.

Pf Since $\varepsilon(f') < \varepsilon(f)$, any $\varepsilon(f)$-fixed arc is also $\varepsilon(f')$-fixed. So need to show some $\varepsilon(f')$-fixed arc that is not $\varepsilon(f)$-fixed. Let p be potentials s.t. f is $\varepsilon(f)$-optimal.

Since f is not minimum cost, let Γ be min mean cost cycle in G_f. Then

$$-\varepsilon(f) = \mu(f) = \frac{c_p(\Gamma)}{|\Gamma|} \geq \frac{-\varepsilon(f)|\Gamma|}{|\Gamma|} = -\varepsilon(f)$$

$\Rightarrow c_p(i,j) = -\varepsilon(f) \ \forall (i,j) \in \Gamma$. No arc $(i,j) \in \Gamma$ is $\varepsilon(f)$-fixed since canceling Γ changes flow on all these arcs, and new circ. is also $\varepsilon(f)$-optimal.

Let ~~$\varepsilon(f)$~~ f' be $\varepsilon(f')$-optimal wrt potentials p'.

Then for Γ, $\frac{c_{p'}(\Gamma)}{|\Gamma|} = \mu(f') = -\varepsilon(f) \leq -2n\varepsilon(f')$

$\Rightarrow \exists (i,j) \in \Gamma$ s.t. $c_{p'}(i,j) \leq -2n\varepsilon(f')$

$\Rightarrow (i,j)$ is $\varepsilon(f')$-fixed and not $\varepsilon(f)$-fixed.

$\blacksquare$