<u>Lemma</u> If we find a blocking flow, then $d(s)$ increases.

<u>Pf</u> Let f, d, ℓ be flow, distances, lengths from prev. iteration
f', d', ℓ' " " " " " next iteration

Want to show for any shortest aug. path P in $G_{f'}$

(1) For all $(i,j) \in P$, $d(i) \le d(j) + \ell'(i,j)$

(2) $\exists (i,j) \in P$ s.t. $d(i) < d(j) + \ell'(i,j)$.

Then $d'(s) = \sum_{(i,j)\in P} \ell'(i,j) > \sum_{(i,j)\in P} (d(i) - d(j)) = d(s) - d(t) = d(s)$.

Showed (1) in class.

For (2), Goldberg & Rao introduce <u>special arc</u> when $d(i) = d(j)$, $\frac{\Delta}{2} \le u_f(i,j) \le \Delta$, $u_f(j,i) \ge \Delta$. Set $\ell(i,j) = 0$. Distances don't change but (i,j) becomes admissible.

Pf of (2): For any path P in $G_{f'}$: $\exists (i,j) \in P$ s.t. either $u_f(i,j) = 0$ or $d(i) < d(j) + l(i,j)$. by properties of blocking flow.

- Spse $u_f(i,j) = 0$ but $(i,j) \in P \Rightarrow u_{f'}(i,j) > 0$

  Must have pushed flow on (j,i)

  $$\Rightarrow d(j) = d(i) + l(j,i)$$

  $$d(i) = d(j) - l(j,i)$$

  Then $d(i) \not< d(j) + l'(i,j)$ if $l'(i,j) = 0$

  $\Rightarrow u_f(i,j) = 0$ and $u_{f'}(i,j) > \Delta$

  $\Rightarrow$ pushed more than Δ flow on (j,i)

- Spse $u_f(i,j) > 0$, $u_{f'}(i,j) > 0$ but $d(i) < d(j) + l(i,j)$

  Then $d(i) \not< d(j) + l'(i,j)$ only if $l(i,j) = 1$, $l'(i,j) = 0$, and $d(i) = d(j)$

$\Rightarrow u_f(i,j) \le \Delta$, $u_{f'}(i,j) > \Delta$, (i,j) not special

because we have $l(i,j) = 1$

$\Rightarrow u_f(i,j) < \frac{\Delta}{2}$

Must have pushed $> \frac{\Delta}{2}$ flow on (j,i)

so that $u_{f'}(i,j) > \Delta$ in next iter.

$\rightarrow\leftarrow$

$\therefore \quad d(i) < d(j) + l'(i,j).$