ORIE 6330 Network Flows October 18, 2012

Problem Set 4

Due Date: November 1, 2012

1. Prove the flow decomposition lemma we mentioned in class. That is, given a circulation f, show that there exist circulations $f_1, \ldots, f_\ell$, for $\ell \leq m$, such that $f = \sum_{i=1}^{\ell} f_i$, $c(f) = \sum_{i=1}^{\ell} c(f_i)$, and for each i, the arcs of f_i with positive flow are a simple cycle.

2. In this problem we will consider another algorithm for the minimum-cost circulation problem, based on the minimum-mean cycle canceling algorithm. Recall from class that we have a minimum-cost circulation f iff G_f has no negative cost cycles. But some negative cost cycles are easier to find than others. For example, consider a cycle in which each edge has negative cost. It can be checked in $O(n)$ time whether there is such a cycle for any cost function c. Call an arc (i, j) *admissible* with respect to potentials p if $c_p(i, j) < 0$ and $u_f(i, j) > 0$. A cycle is admissible if it consists entirely of admissible arcs. Consider Algorithm 1.

   (a) We need to give a procedure for updating the potentials in step 4 of the algorithm. Recalling the proof that $\mu(f) = -\epsilon(f)$, give such a procedure.

   (b) Prove the following claim: when updating the potentials, $\epsilon(f)$ has decreased by a factor of $(1 - 1/n)$ since the last update.

   (c) Prove the following claim: in each pass through step 3, at most m cycles are cancelled.

   (d) Prove the following claim: at most $O(n \log(nC))$ iterations of the while loop of step 2 are needed to obtain an optimal circulation f.

   (e) Prove that the overall running time of the algorithm is $O(mn^2 \log(nC))$.

3. In class, we discussed a subroutine for converting a 2ϵ-optimal circulation to an ϵ-optimal circulation via a push/relabel algorithm, which resulted in an $O(n^2 m \min(\log(nC), m \log n))$ time algorithm for the minimum-cost circulation problem. In this problem, we will give a subroutine for the same problem based on blocking flows. Consider the subroutine in Algorithm 2. Below we let G_A be the graph of currently admissible arcs (that is, $c_p(i, j) < 0$ and $u_f(i, j) > 0$).

1 Initialize potentials p and find a feasible circulation f
2 **while** f is not a minimum-cost circulation **do**
3   Repeatedly find and cancel admissible cycles until none exist
4   Update potentials p so that $c_p(i, j) \geq -\epsilon(f)$ for all $(i, j) \in A_f$
5 **return** f

Algorithm 1: Canceling admissible cycles.

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for (i, j) ∈ A do
    if c_p(i, j) < 0 then
        f(i, j) ← u(i, j)
while f is not a circulation do
    S ← {i ∈ V : ∃j ∈ V such that e_f(j) > 0, i reachable from j in G_A}
    forall the i ∈ S do p_i ← p_i − ε
    Form network N from G_A by adding source s, sink t, arc (s, i) of capacity e_f(i)
            for all i ∈ V with e_f(i) > 0, arc (i, t) of capacity e_f(i) for all i ∈ V with
            e_f(i) < 0
    Find blocking flow b on N
    f ← f + b
return (f, p)
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Algorithm 2: Blocking flow subroutine

Given a blocking flow algorithm that runs in $O(m \log n)$ time, prove that the subroutine above is correct and runs in $O(mn \log n)$ time. This gives a $O(mn \log n \min(\log(nC), m \log n))$ time algorithm for the minimum-cost circulation problem.

4. This problem generalizes the minimum mean-cost cycle problem. Let $G = (V, A)$ be a directed graph, with integer costs $c(i,j)$ and time $t(i,j) \geq 0$ for each $(i,j) \in A$. Assume that for every cycle Γ, $t(\Gamma) = \sum_{(i,j)\in\Gamma} t(i,j) > 0$. Let $T = \max_{(i,j)\in A} t(i,j)$ and $C = \max_{(i,j)\in A} |c(i,j)|$. Give a $O(mn \log(nCT))$ time algorithm for finding a cycle that minimizes
$$\min_{\text{cycles } \Gamma \in G} \frac{c(\Gamma)}{t(\Gamma)}.$$
Note that in the case $t_{ij} = 1$ for all $(i,j) \in A$, this is just the problem of finding a minimum mean-cost cycle.