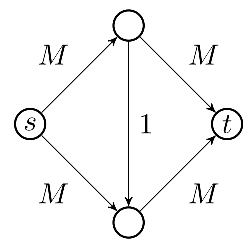
ORIE 6330 Network Flows August 30, 2012

Problem Set 1

Due Date: September 13, 2012

1. Consider the maximum s-t flow problem shown below. Show that an algorithm that chooses arbitrary augmenting paths from the residual graph does not run in polynomial time.



2. Consider the following image segmentation problem. We are given as input an undirected graph $G = (V, E)$, labels L, assignment costs $c(i, \ell) \geq 0$ for all $i \in V$ and $\ell \in L$, and separation costs $p(i, j) \geq 0$ for all $(i, j) \in E$. We would like to assign each vertex a single label of the graph. There is a cost $c(i, \ell)$ for assigning label ℓ to vertex $i \in V$. Also, all things being equal, we want nearby vertices assigned the same labels, so there is a penalty $p(i, j)$ for edge $(i, j) \in E$ if i and j are assigned different labels. The goal is to find a labeling that minimizes the overall cost (assignment costs plus penalties). More formally, we want to find a labeling of the vertices $f : V \to L$ that minimizes
$$\sum_{i \in V} c(i, f(i)) + \sum_{(i,j) \in E : f(i) \neq f(j)} p(i, j).$$
Show that if there are only two labels (that is, $|L| = 2$), then we can find the labeling of minimum cost via a minimum s-t cut computation.

3. In this problem, we return to the baseball elimination problem described in class. We use the same notation as was used in class.

    (a) Prove that if team k is eliminated, and $w(k) + g(k) \geq w(i) + g(i)$, then team i is also eliminated (Note: what did we use to prove that team k is eliminated? How can this be extended to team i?).

    (b) Prove that we can determine which teams in the division have been eliminated and which have not using with $O(\log |T|)$ executions of a maximum flow algorithm, where T is the set of teams in the division.

4. Let $G = (V, A)$ be a directed graph, with bounds $0 \leq l(i, j) \leq u(i, j)$ for all $(i, j) \in A$. We say that f is a feasible *circulation* if for all $i \in V$,
$$\sum_{j:(j,i) \in A} f(j, i) - \sum_{j:(i,j) \in A} f(i, j) = 0,$$

and for all $(i,j) \in A$, $l(i,j) \leq f(i,j) \leq u(i,j)$.

For $S \subset V$, recall that $\delta^+(S) = \{(i,j) \in A : i \in S, j \notin S\}$ and $\delta^-(S) = \{(i,j) \in A : i \notin S, j \in S\}$. Also, for a subset $A' \subseteq A$ of arcs, define $u(A') = \sum_{(i,j)\in A'} u(i,j)$ and $l(A') = \sum_{(i,j)\in A'} l(i,j)$.

Prove that there exists a feasible circulation f if and only if for all subsets $S \subset V$, $S \neq \emptyset$, $u(\delta^+(S)) \geq l(\delta^-(S))$.