ORIE 633 Network Flows October 18, 2007

Lecture 13

Lecturer: David P. Williamson *Scribe: Hyung-Chan An*

Recall from the previous lecture the following definitions and algorithm:

Definition 1 *The mean cost of cycle* Γ *is* $\dfrac{c(\Gamma)}{|\Gamma|} = \dfrac{c^p(\Gamma)}{|\Gamma|}$.

Definition 2 *For a circulation* f*,*

$$\mu(f) = \min_{\text{cycles } \Gamma \text{ in } G_f} \frac{c(\Gamma)}{|\Gamma|}.$$

Min mean-cost cycle cancelling (Goldberg, Tarjan '89)

Let f be any feasible circulation
While $\mu(f) < 0$
  Find a min mean-cost cycle Γ
  Cancel Γ; update f

In order to analyze this algorithm, we introduced the notion of ϵ-optimality:

Definition 3 *A circulation* f *is* ϵ*-optimal if there exist potentials* p *such that* $c^p_{ij} \geq -\epsilon$ *for all* $(i,j) \in A_f$.

Theorem 1 *A* 0*-optimal circulation is optimal.*

Definition 4 *Let* $\epsilon(f)$ *be the minimum* ϵ *such that* f *is* ϵ*-optimal.*

Let $f^{(k)}$ denote the circulation obtained by the algorithm after k iterations. The following theorems were proven in the previous lecture.

Theorem 2 $\epsilon(f^{(1)}) \leq \epsilon(f)$.

Theorem 3 $\epsilon(f^{(m)}) \leq \left(1 - \dfrac{1}{n}\right)\epsilon(f)$.

Theorem 4 *If* $\epsilon(f) < \frac{1}{n}$*, then* f *is optimal.*

Theorem 5 *Any circulation is* C*-optimal, where* $C = \max|c_{ij}|$.

Theorem 6 $O(mn\log(nC))$ *iterations of the algorithm result in an optimal circulation; hence the given algorithm runs in* $O(m^2n^2\log(nC))$ *time.*

1 Strongly polynomial-time analysis of the min mean-cost cycle cancelling algorithm

In this lecture, we present a strongly polynomial running time analysis of the min mean-cost cycle cancelling algorithm described above. This analysis is based on Professor Éva Tardos's algorithm and analysis, with modifications to fit the given algorithm. It was a significant progress to answer the question whether we can have a strongly polynomial-time algorithm to solve this class of linear programs; she was awarded Fulkerson prize for this work.

Definition 5 *An arc (i, j) is ϵ-fixed if the flow on it is the same for all ϵ-optimal circulations.*

Intuitively, we will show that the arcs are 'fixed' one by one throughout the execution of the algorithm, and that the arcs, once fixed, remain fixed until the termination. We use the revised definition of a circulation given in the previous lecture that introduces antisymmetry.

Lemma 7 *For any circulation f and any $S \subsetneq V$ $(S \neq \emptyset)$,*

$$\sum_{\substack{k \in S \\ l \notin S \\ (k,l) \in A}} f_{kl} = 0.$$

Proof: By flow conservation, $\sum_{l:(k,l)\in A} f_{kl} = 0$ for any k. Summing over S,

$$\sum_{k \in S} \sum_{l:(k,l)\in A} f_{kl} = 0.$$

Since $f_{kl} + f_{lk} = 0$, all the terms in the left-hand side with $k, l \in S$ cancel out; thus,

$$\sum_{\substack{k \in S \\ l \notin S \\ (k,l) \in A}} f_{kl} = 0.$$

□

The following theorem shows a condition for an arc to be ϵ-fixed.

Theorem 8 *For $\epsilon > 0$, let f be a circulation and p be potentials such that f is ϵ-optimal with respect to p. If $|c^p_{ij}| \geq 2n\epsilon$, then (i, j) is ϵ-fixed.*

Proof: Proof by contradiction. Let f' be an ϵ-optimal circulation such that $f'_{ij} \neq f_{ij}$. Assume that $c^p_{ij} \leq -2n\epsilon$; this assumption is without loss of generality since the costs are antisymmetric. Then we claim:

Claim 9 *There exists a cycle Γ in $A_{f'}$ containing (i, j).*

Proof: Since $c^p_{ij} \leq -2n\epsilon$ and f is ϵ-optimal with respect to p, $(i,j) \notin A_f$. Hence $f_{ij} = u_{ij}$ and $f'_{ij} < f_{ij}$.

Let $E_< = \{(k,l) \in A : f'_{kl} < f_{kl}\}$. Then $E_< \subseteq A_{f'}$, since $f'_{kl} < f_{kl} \leq u_{kl}$. Thus if we find a cycle Γ in $E_<$, then Γ is in $A_{f'}$.

Let S be the set of the nodes reachable from j in $E_<$. We will show that $i \in S$; this will imply that there exists a cycle Γ including (i,j).

Suppose $i \notin S$. We know

$$\sum_{\substack{k \in S \\ l \notin S \\ (k,l) \in A}} \left(f_{kl} - f'_{kl}\right) = 0$$

from Lemma 7; $f'_{ij} < f_{ij}$ yields $f_{ji} - f'_{ji} < 0$; hence, there exists (k,l) such that $k \in S$, $l \notin S$ and $f_{kl} - f'_{kl} > 0$ (see Figure 1). However, $f_{kl} - f'_{kl} > 0$ implies $(k,l) \in E_<$ and hence $l \in S$, leading to contradiction.

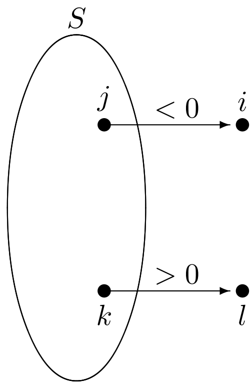

Figure 1: $f_{ji} - f'_{ji} < 0 \quad \Longrightarrow \quad \exists (k,l) \in \delta^+(S) \;\; f_{kl} - f'_{kl} > 0.$

(*A quick alternative proof by Yogi:* observe that $f - f'$ is a circulation and $f_{ij} - f'_{ij} > 0$. Since every circulation can be decomposed into at most m cycles as shown in the problem set, there exists a cycle Γ that has $(i,j) \in \Gamma$ in $E_<$.) □

Now back to the proof of Theorem 8. We know that there exists a cycle $\Gamma \subseteq A_{f'}$ with $(i,j) \in \Gamma$. Both proofs further show that $\Gamma \subseteq E_<$.

For all $(k,l) \in \Gamma$, $f_{lk} < f'_{lk} \leq u_{lk}$ since $\Gamma \subseteq E_<$ and f is antisymmetric; hence, $(l,k) \in A_f$. Therefore $c^p_{lk} \geq -\epsilon$, yielding $c^p_{kl} \leq \epsilon$. Thus,

$$\begin{aligned} \mu(f') = \frac{c(\Gamma)}{|\Gamma|} = \frac{c^p(\Gamma)}{|\Gamma|} &= \frac{1}{|\Gamma|}\left(c^p_{ij} + \sum_{\substack{(k,l) \in \Gamma \\ (k,l) \neq (i,j)}} c^p_{kl}\right) \\ &\leq \frac{1}{|\Gamma|}(-2n\epsilon + (|\Gamma| - 1)\epsilon) \\ &< -\epsilon. \end{aligned} \tag{1}$$

We already know $\epsilon \geq \epsilon(f') = -\mu(f')$, which contradicts (1).
Therefore, the flow on (i,j) is fixed. $\square$

Now we are ready to analyze the running time.

Theorem 10 *The min mean-cost cycle cancelling algorithm terminates after $O(m^2 n \log n)$ iterations.*

Proof: Once an arc is fixed, it remains fixed for the rest of the algorithm since $\epsilon(f)$ is non-increasing. The following claim completes the proof. $\square$

Claim 11 *A new arc is fixed after $k = mn\log(2n)$ iterations.*

Proof: Let f be the current circulation and Γ be the cycle cancelled in this iteration.
By Theorem 3,

$$\epsilon(f^{(k)}) \leq \left(1 - \frac{1}{n}\right)^{n\log(2n)} \epsilon(f) < \frac{\epsilon(f)}{2n}.$$

Let p^k be the node potentials for circulation $f^{(k)}$ such that $f^{(k)}$ is $\epsilon(f^{(k)})$-optimal. Then

$$\frac{c^{p^k}(\Gamma)}{|\Gamma|} = \mu(f) = -\epsilon(f) < -2n\epsilon(f^{(k)});$$

hence, there exists $(i,j) \in \Gamma$ such that $c_{ij}^{p^k} < -2n\epsilon(f^{(k)})$. Therefore, (i,j) is $\epsilon(f^{(k)})$-fixed.
Note that (i,j) was not $\epsilon(f)$-fixed because $(i,j) \in \Gamma$ and the flow on it was changed when we canceled Γ. $\square$

In the next lecture, we will present another type of algorithm for the min-cost circulation problem that is more efficient.