In class, we saw that every bounded polyhedra is a polytope in the set of convex combination of its vertices. Now we will extend the theory to pointed polyhedra (i.e., those that contain no lines).

**Definition 1** Let \( C \) be a nonempty convex set: then the recession cone of \( C \), \( \text{rec}(C) \), is
\[
\{ d \in \mathbb{R}^m : \forall x \in C, \forall \alpha \geq 0, x + \alpha d \in C \}.
\]

**Proposition 1** If \( C \) is a nonempty set then \( \text{rec}(C) \) is a nonempty convex cone.

**Proof:** Let \( d_1, d_2 \in \text{rec}(C), \lambda_1, \lambda_2 \geq 0 \). We want to show that \( \lambda_1 d_1 + \lambda_2 d_2 \in \text{rec}(C) \). For any \( x \in C \) and any \( \alpha \geq 0 \)
\[
x + \alpha (\lambda_1 d_1 + \lambda_2 d_2) = [x + (\alpha \lambda_1) d_1] + (\alpha \lambda_2) d_2.
\]
The quantity in brackets lies in \( C \) since \( \alpha \lambda_1 \geq 0 \) and \( d_1 \in \text{rec}(C) \), and then the desired vector lies in \( C \) because \( \alpha \lambda_1 \geq 0 \) and \( d_2 \in \text{rec}(C) \). Also, \( 0 \in \text{rec}(C) \) by definition.

**Proposition 2** For \( Q := \{ y \in \mathbb{R}^m : A^T_x y \leq c_x, A^T_w y = c_w \} \) then (if \( Q \) is nonempty)
\[
\text{rec}(Q) = \{ d \in \mathbb{R}^m : A^T_x d \leq 0, A^T_w d = 0 \}.
\]

**Proof:**
\[\supseteq:\]
if \( A^T_x d \leq 0, A^T_w d = 0 \) then for any \( y \in Q, \alpha \geq 0 \)
\[
A^T_x (y + \alpha d) = A^T_x y + \alpha A^T_x d \leq c_x + 0 = c_x,
\]
Similarly \( A^T_w (y + \alpha d) = c_w \),

hence \( (y + \alpha d) \in Q \).

\[\subseteq:\]
Suppose \( d \in \text{rec}(Q) \), and choose any \( y \in Q \). Then \( \forall \alpha \geq 0 \)
\[
A^T_x (y + \alpha d) = A^T_x y + \alpha A^T_x d \leq c_x,
\]
then \( A^T_x y \leq c_x \)
\[
A^T_x d \leq 0.
\]
otherwise, the above would fail for large \( \alpha \)
similarly, \( A^T_w d = 0 \).

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1Based on script of Fall 2005 lecture 5 by Gurmeet Singh
Theorem 3 (Representation of Pointed Polyhedra). Let $Q$ (defined as in Proposition 2) be a nonempty pointed polyhedron, and let $P$ be the set of all convex combinations of its vertices and $K$ be its recession cone. Then

$$Q = P + K := \{p + d : p \in P, d \in K\}.$$

Proof:

$\supseteq$:
Every vertex of $Q$ satisfies all linear constraints of $Q$ so $p$ also does for any $p \in P$.
So any $p + d \in P + K$ has

$$A^T_x(p + d) = A^T_xp + A^T_yd \leq c_x + 0 = c_x;$$
$$A^T_w(p + d) = A^T_wp + A^T_yd = c_w + 0 = c_w.$$

$\subseteq$:
The proof is by induction on $\{m - ra(y)\}$.

True for $\{m - ra(y) = 0\} \iff y$ is itself a vertex of $Q$ and $d = 0 \in \text{rec}(C)$.

Suppose true if $\{m - ra(y) < k\}$ for some $k > 0$ and consider $y \in Q$ with $ra(y) = m - k < m$. Choose $0 \neq d \in \mathbb{R}^m$ with $a^T_jd = 0, \forall j \in I(y)$ and consider $y + \alpha d, \alpha \in \mathbb{R}$. Since $Q$ is pointed there are three cases to consider.

(1) $\alpha$ is bounded above and below, say by $\underline{\alpha} < \overline{\alpha}$.

As in the previous theorem

$$y = \frac{\overline{\alpha}}{\overline{\alpha} - \underline{\alpha}}(y + \alpha d) + \frac{-\alpha}{\overline{\alpha} - \underline{\alpha}}(y + \alpha d),$$
and $(y + \alpha d)$ has $m - ra(y + \alpha d) < k$, so

$$(y + \alpha d) = p + \bar{d} , \ p \in P , \ \bar{d} \in K,$$
and similarly

$$(y + \overline{\alpha} d) = p + d , \ p \in P , \ d \in K,$$
so

$$y = \frac{\overline{\alpha}}{\overline{\alpha} - \underline{\alpha}}p + \frac{-\alpha}{\overline{\alpha} - \underline{\alpha}}(p + \bar{d})
= \frac{\overline{\alpha}}{\overline{\alpha} - \underline{\alpha}}[p + \frac{-\alpha}{\overline{\alpha} - \underline{\alpha}}(p + \bar{d})]
+ \{...d + ...d\}.$$

The vector in brackets is a point of $P$ and that in braces a point in $K$.

(2) $\alpha$ is bounded below but not above. Then $d \in K$ and $y = [y + \alpha d] + (-\alpha)d$, with $\alpha$ defined as before. The vector in brackets lies in $P + K$ as in the first part by the inductive hypothesis. Therefore

$$y = (p + d) + (-\alpha)d$$
$$= \frac{\overline{\alpha}}{\overline{\alpha} - \underline{\alpha}}[p + \frac{-\alpha}{\overline{\alpha} - \underline{\alpha}}(p + \bar{d})] + \{...d + ...d\}.$$
lies in $P + K$.

(3) $\alpha$ is bounded above but not below. Then we can simply switch $d$ to $-d$ and $\sigma$ to $-\sigma$, and we get back to case (2).

This completes the proof.

\[ \square \]

**Theorem 4** (Fundamental theorem of LP). Consider the LP problem $\max \{b^T y : y \in Q\}$ with $Q$ being a pointed polyhedron. Then

1. if there is a feasible solution, there is a vertex solution (basic feasible solution);
2. if there is a feasible solution and $b^T y$ is unbounded above on $Q$, then there is a ray or halfline: 
   \begin{align*}
   \{y + \alpha d : \alpha \geq 0\} \in Q \text{ on which } b^T y \text{ is unbounded above; and}
   \end{align*}
3. if $b^T y$ is bounded above on $Q$, then the max is attained and attained at a vertex $Q$.

**Proof:**

(1): If $Q \neq \emptyset, P \neq \emptyset$, so there exists a vertex.

(2) & (3):

Assume $P \neq \emptyset$ & $P$ is a set of convex combinations of $v_1, v_2, v_3, ..., v_k$.

\[
\sup\{b^T y : y \in Q\} = \sup\{b^T y : y \in P + K\} = \sup\{b^T p + b^T d : p \in P, d \in K\} = \sup\{b^T p : p \in P\} + \sup\{b^T d : d \in K\}.
\]

If there is some $\overline{d} \in K$ with $b^T \overline{d} > 0$ then by considering $\alpha \overline{d}, \alpha \to +\infty$, see that $\sup\{b^T d : d \in K\} = +\infty$. Then $b^T y$ is unbounded above on $Q$ and clearly unbounded above on $\{y + \alpha \overline{d} : \alpha \geq 0\}$ for any $y \in Q$.

If there is no such $\overline{d} \in K$, then $\sup\{b^T d : d \in K\} = 0$, attained by $d = 0$. Then

\[
\sup\{b^T y : y \in Q\} = \sup\{b^T p : p \in P\} = \sup\{\sum_{i=1}^{k} \lambda_i (b^T v_i) : \sum_{i=1}^{k} \lambda_i = 1, \text{ all } \lambda_i \geq 0\} = \max_{1 \leq i \leq k} b^T v_i.
\]

In this case $\sup\{b^T y : y \in Q\}$ is attained by $y = v_i$ where $i$ attains the maximum.

\[ \square \]