ORIE 6300 Mathematical Programming I	September 17, 2008
Recitation 3	
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Feasibility and Unboundedness [1]

Consider a linear program in arbitrary form. We know that it can potentially be infeasible or have unbounded optimal objective. Additionally, if it's feasible and not unbounded, we can show the existence of an optimal solution by applying the Weierstrass Theorem. Hence, this gives three options for the types of solutions a linear program can have.

Additionally, the dual of a linear program is itself a linear program, so the same three options apply. Hence, the first natural question is what combinations of these can appear for a primal-dual pair of linear programs?

$P\backslash D$	I	O	U
I	?	?	?
O	?	?	?
U	?	?	?

Let's try to fill in some of these boxes.

First, recall the weak duality theorem: If x is a feasible solution to a minimization linear program and y is a feasible solution to its dual, then $b^T y \leq cx$.

Suppose the primal minimization program is unbounded. This immediately implies that the dual must be infeasible. Similarly, if the dual is unbounded, this immediately implies that the primal must be infeasible. To see this in the first case, let y be any feasible solution to the dual. Since the primal is unbounded, there exists an $\hat{x}$ such that $c\hat{x} < b^T y$, contradicting the Weak Duality Theorem. Hence, no such y exists. The other argument can be proved identically.

Hence, our table now looks like:

$P\backslash D$	I	O	U
I	?	?	✓
O	?	?	X
U	✓	X	X

Given the above theorem, it seems natural to ask whether the reverse implication holds. Does primal infeasibility imply dual unboundedness? Consider the following LP:

[1] Based on previous notes of Rajithkumar Rajagopalan and Dennis Leventhal

$$\begin{array}{ll} \max & [2, -1]\ x \\ \text{s.t.} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x \le \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ & x \ge 0 \end{array}$$

Its corresponding dual is:

$$\begin{array}{ll} \max & [-1, 2]\ y \\ \text{s.t.} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} y \le \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ & y \ge 0 \end{array}$$

Note that the primal is infeasible and that the dual feasible region is exactly the primal feasible region, hence, both are infeasible. This adds another option to our table, giving:

Finally, using Strong Duality Theorem we know when one of primal or the dual has an optimal solution, they both must have an optimal solution. Hence our table looks like:

$P \backslash D$	I	O	U
I	✓	X	✓
O	X	✓	X
U	✓	X	X

Farkas Lemma and its Application

First recall the Farkas' Lemma :

Theorem 1 (Farkas' Lemma) *If $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then exactly one of the following holds:*

1. $\exists x \ge 0$ *such that* $Ax \le b$
2. $\exists y$ *such that* $A^T y \ge 0, b^T y < 0, y \ge 0$

Here is another form of the Farkas Lemma:

Theorem 2 (Alternative Farkas' Lemma) *Exactly one of the following holds:*

1. $\exists x \ge 0$ *such that* $Ax \le b$
2. $\exists y$ *such that* $A^T y \ge 0, b^T y < 0, y \ge 0$

Proof: Note $Ax = b$, $x \ge 0$ is feasible if and only if $Ax + s = b$, $x, s \ge 0$ is feasible. Apply the original Farkas Lemma to this new system. □

Let's see an application of the Farkas Lemma. Note that we can only prove that unboundedness implies infeasibility for linear programs and not the converse in the previous section. We now prove a related implication for the unboundedness of feasible **regions** of linear programs.

Theorem 3 (Clark's Theorem) *Given the following primal and dual LPs, if one of them is feasible, then the feasible region for one of them is non-empty and unbounded.*

Primal LP:		*Dual LP:*	
min	cx	*max*	yb
s.t.:	$Ax \geq b$	*s.t.:*	$yA \leq c$
	$x \geq 0$		$y \geq 0$

It's important to note that the result of the theorem is that the feasible region of one of the LPs is unbounded, but it may not be the case that the LP has unbounded objective function value with the given objective function.

Proof: There are three possibilities to consider.

1. The primal is infeasible and the dual is unbounded.
2. The dual is infeasible and the primal is unbounded.
3. Both the primal and dual are feasible and not unbounded (hence have optimal solution).

In the first two cases, we immediately have the result we want. Hence, suppose we're in the last case. Now, consider a different primal problem with c replaced by $\hat{c} = [-1, -1, \ldots, -1]$ and its associated dual.

Primal LP:		Dual LP:	
min	$\hat{c}x$	max	$b^T y$
s.t.:	$Ax \geq b$	s.t.:	$A^T y \leq \hat{c}$
	$x \geq 0$		$y \geq 0$

If this new LP has unbounded objective function value, then the primal has an unbounded feasible region, and we're done. Otherwise, the primal has bounded objective function value. Note that the modified primal has the same feasible region as the original primal, hence it is feasible by assumption. Then, we can apply the Alternative Farkas' lemma to get the following system in which one and only one can hold:

1. $\exists y$ such that $A^T y \leq \hat{c}, y \geq 0$
2. $\exists x$ such that $Ax \geq 0, \hat{c}^T x < 0,\ x \geq 0$

If (2) holds, let $\hat{x}$ be a feasible solution to (2) and x be a feasible solution to modified LP and λ ¿ 0. Then

$$A(x + \lambda\hat{x}) = Ax + \lambda A\hat{x} \geq b + \lambda * 0 = b$$

So $\lambda\hat{x}$ is feasible for all $\lambda \geq 0$. Additionally, $\lambda c\hat{x} \to -\infty$ as $\lambda \to \infty$, so the modified primal which has the same feasible region as the original primal is unbounded and we are done.

Otherwise if (1) holds, let $\hat{y}$ be a feasible solution to the modified dual and let y be a feasible solution to the original dual. Then using a similar argument as above, we can show that for any $k \geq 0$, $y + k\hat{y}$ is feasible for the original dual:

First, we know that it satisfies $y + k\hat{y} \geq 0$ since $y \geq 0$, $\hat{y} \geq 0$, and $k \geq 0$. Now, we need to show that $A^T(y + k\hat{y}) \leq c$. We know that $A^T y \leq c$, and $A^T\hat{y} \leq \hat{c}$. But since $\hat{c}$ is the vector of -1s, that gives

$$kA^T\hat{y} \leq k\hat{c} \leq 0.$$

Thus, we have that $A^T(y + k\hat{y}) \leq c$ for any $k \geq 0$.

Thus, the direction $\hat{y}$ give us an unbounded direction in the feasible region for the original dual. Hence in both cases we have at least one of the primal or the dual having an unbounded feasible region, and we're done $\square$