ORIE 6300 Mathematical Programming I November 8, 2008

Problem Set 9

Due Date: November 14, 2008

1. (23 points) At the end of the lecture on November 6, we considered the n-dimensional unit ball $E_0 = \{x \in \Re^n : \sum_{i=1}^n x_i^2 \leq 1\}$, and considered cutting it through the center with the hyperplane $x_1 = 0$. We wanted to show that the ellipsoid

$$E_1 = \left\{ x \in \Re^n : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^n x_i^2 \right\}$$

was a good example for the ellipsoid method in that it contains the region $E_0 \cap \{x \in \Re^n : x_1 \geq 0\}$ and that the volume of E_1 is measurably less than that of E_0. In this problem, you will prove both statements.

   (a) (7 points) Show that $E_0 \cap \{x \in \Re^n : x_1 \geq 0\} \subseteq E_1$.

   (b) (3 points) Show that $1 + x \leq e^x$ for all values of x.

   (c) (5 points) Ellipsoids take the form $\{x \in \Re^n : (x-z)^T B^{-1}(x-z) \leq 1\}$, where z is the center of the ellipsoid and B is symmetric and positive definite. Show that E_1 has this form (though you need not show that B is positive definite).

   (d) (8 points) The volume of an ellipsoid given in the form of the part above is $\sqrt{\det(B)}$ times the volume of a unit ball. Use this to show that the volume of E_1 is at most $e^{-1/2(n+1)}$ times the volume of E_0.

2. (35 points) In recitation on November 5, you discussed the minimum-cost spanning tree problem and showed that a particular algorithm gives an optimal solution to the problem by giving a dual solution to a linear programming relaxation of the problem that has the same value. Here we will see another linear programming relaxation of the problem also works.

We could consider a formulation similar to that we used for the traveling salesman problem. In that formulation, we considered sets of edges $\delta(S) = \{(u,v) \in E : u \in S, v \notin S\}$, where $S \subset V$. Consider the following linear program:

$$\begin{array}{lrcll} \text{Min} & \displaystyle\sum_{e \in E} c_e x_e & & & \\ & \displaystyle\sum_{e \in \delta(S)} x_e & \geq & 1 & \forall S \subset V, S \neq \emptyset \\ & x_e & \geq & 0 & \forall e \in E. \end{array}$$

   (a) (5 points) Argue that this is a linear relaxation of the minimum-cost spanning tree problem; that is, given an minimum-cost spanning tree T, if we set $x_e = 1$ for all $e \in T$, this is a feasible solution for the LP that has no greater cost than T.

   (b) (5 points) Show that there are graphs for which the LP has cost strictly smaller than that of the optimal spanning tree (Hint: there is an example on three vertices).

Now let $\mathcal{P}$ be a partition of the vertex set; i.e. $\mathcal{P} = \{P_1, \ldots, P_k\}$ such that $P_i \cap P_j = \emptyset$ if $i \neq j$ and $\bigcup_i P_i = V$. Given any $\mathcal{P}$, if we sum together the constraints of the LP above associated with $P_i \in \mathcal{P}$, we have that

$$\sum_{P_i \in \mathcal{P}} \sum_{e \in \delta(P_i)} x_e \geq |\mathcal{P}|.$$

Yet we can say something stronger than this.

(c) (5 points) Prove that for any spanning tree T, if $x_e = 1$ for all $e \in T$, then for any partition $\mathcal{P}$,

$$\sum_{P_i \in \mathcal{P}} \sum_{e \in \delta(P_i)} x_e \geq 2|\mathcal{P}| - 2.$$

Now consider the following linear program.

$$\begin{aligned}
\text{Min} \sum_{e \in E} c_e x_e & \\
\sum_{P_i \in \mathcal{P}} \sum_{e \in \delta(P_i)} x_e &\geq 2|\mathcal{P}| - 2 \qquad \forall \text{ partitions } \mathcal{P} \\
x_e &\geq 0 \qquad \forall e \in E.
\end{aligned}$$

(d) (5 points) Argue that the linear program above is a linear programming relaxation of the minimum-cost spanning tree problem.

(e) (15 points) Using the same minimum-cost spanning tree algorithm that you did in class, show that the algorithm produces the optimal solution by using the linear programming relaxation above and its dual. This also proves that the linear program always returns a solution of cost equal to the minimum-cost spanning tree.