ORIE 6300 Mathematical Programming I October 24, 2008

Problem Set 7

Due Date: October 31, 2008

1. In class, we started to describe the capacitated simplex method for solving linear programs of the form $\min c^T x : Ax = b, \ell \le x \le u$. We noted that its dual is $\max b^T y - u^T v + \ell^T w : A^T y - v + w = c$, and that for any y, we can have a dual feasible solution by setting $v = \max(0, A^T y - c)$ and $w = \max(0, c - A^T y)$. We stated that the main idea is now to maintain three sets of indices of the primal variables: B the basic variables, L the variables set to their lower bounds, and U the variables set to the upper bounds, so that associated with these sets, we have a solution x in which we set $x_j = \ell_j$ for all $j \in L$, $x_j = u_j$ for all $j \in U$, and $x_B = A_B^{-1}(b - A_U u_U - A_L \ell_L)$; we assume the primal is feasible, which is true if $\ell_B \le x_B \le u_B$.

   Given the dual solution $y = (A_B^T)^{-1} c_B$, and the normal reduced costs $\bar{c} = c - A^T y$, we argued in class that the current primal and dual are optimal if $\bar{c}_j \ge 0$ for all $j \in L$ and $\bar{c}_j \le 0$ for all $j \in U$. Finish the description of the simplex method by describing what should happen from this point on: if the solutions are not optimal, how should x, B, L, and U be altered so that x remains feasible and so that we make progress if the current solution is not degenerate? How do we know that the updated B is a basis?

2. In the recitation on October 15, you considered the assignment problem. The assignment problem can be formulated as the following integer program:

$$\begin{aligned} \text{Min } & \sum_{i,j} c_{ij} x_{ij} \\ & \sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, \ldots, n \\ & x_{ij} \in \{0, 1\} \qquad i = 1, \ldots, n; j = 1, \ldots, n. \end{aligned}$$

   If we replace the integrality conditions $x_{ij} \in \{0,1\}$ by $x_{ij} \ge 0$, we have a linear program. The dual of that linear program is

$$\begin{aligned} \text{Max } & \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \\ & u_i + v_j \le c_{ij} \qquad i = 1, \ldots, n; j = 1, \ldots, n. \end{aligned}$$

   The goal of this problem is to have you reconsider the Hungarian algorithm given in the recitation in terms of these primal and dual linear programs. Let $\bar{c}_{ij} = c_{ij} - u_i - v_j$, be the *reduced cost* of assigning worker i to job j. We suppose that $c \ge 0$ and c is integral. Initially we set $u = 0$ and $v = 0$. Set $u_i = \min_{j=1,\ldots,n} c_{ij}$ for each i, and then $v_j = \min_{i=1,\ldots,n} \bar{c}_{ij}$ for $j = 1, \ldots, n$.

Now translate the rest of the Hungarian algorithm into this framework; i.e. restate the Hungarian algorithm as given in the recitation into an algorithm that solves the primal and dual linear programs given above. What is a 0-cover? What change in the dual corresponds to the operations which we perform when we have a 0-cover of less than n lines? How do we know we make progress at every step? How do we know the algorithm will terminate in a finite number of iterations?

As a final question: when we replace the integrality conditions $x_{ij} \in \{0, 1\}$ by $x_{ij} \geq 0$, does the value of the optimal solution to the program change? In which direction? Why?