ORIE 6300 Mathematical Programming I September 26, 2008

Problem Set 4

Due Date: October 3, 2008

1. In class we showed that given a polytope $Q = conv(v_1, \ldots, v_k)$, if 0 is in the interior of Q, then Q is a bounded polyhedron. Now suppose that we only know that there is some point v in the interior of Q. Show that Q is bounded polyhedron. (Hint: Think about $Q - v = \{w - v : w \in Q\}$).

2. Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron. Show that if 0 is on the boundary of P (that is, $0 \in P$, but 0 is not in the interior of P), then the polar P^* of P is not bounded.

3. We proved strong duality using Farkas' Lemma. Suppose that we had proved strong duality in some other way. Use strong duality to prove that Farkas' Lemma is true.

4. (Strict Complementary Slackness) Consider the standard form linear programs, with primal LP $(\min c^T x : Ax = b, x \geq 0)$ and dual LP $(\max b^T y : A^T y \leq c)$. Suppose the value of the two LPs is γ.

   (a) Show that the set of optimal solutions to the primal is a convex set; argue the same for the dual.

   (b) Show that either there exists an optimal solution x to the primal such that $x_j > 0$ or there exists an optimal solution y to the dual such that the jth inequality is strict; that is, $\sum_{i=1}^{n} a_{ij} y_i < c_j$. (Hint: Consider the linear program $(\min -e_j^T x : Ax = b, -c^T x \geq -\gamma, x \geq 0)$, where e^j is a vector that has a 1 in the jth component, and 0 everywhere else).

   (c) Show that there exist a primal optimal solution x^* and a dual optimal solution y^* such that $x_j^* > 0$ if and only if the jth inequality of the dual is met with equality.