ORIE 6300 Mathematical Programming I September 11, 2008

Problem Set 2

Due Date: September 19, 2008

1. Recall the maximum flow problem, and its dual as they were presented in class. We used the variables z^*_{uv} of an optimal dual solution to define $cost(s, v)$ for each vertex v, and then defined the sets $S_\rho = \{v : cost(s, v) \leq \rho\}$, and showed that each S_ρ is an s-t cut for $0 \leq \rho < 1$. We have also shown that, in any optimal dual solution z^*, at least one of the cuts S_ρ for some $0 \leq \rho < 1$ defines a minimum cut by showing that the expected value $E[n(S_\rho)] \leq \sum_{(u,v)\in A} z^*_{uv}$.

   (a) Show that all cuts S_ρ occurring with positive probability in the expectation must be minimum s-t cuts for the graph.

   (b) (Extra credit) Show that all cuts S_ρ cuts are minimum s-t cuts for the graph.

2. (Carathéodory's theorem) Show that if $x \in \Re^n$ is a convex combination of $v_1, \ldots, v_k$, then it is also a convex combination of at most $n + 1$ of these points.

3. Consider the polytope $Q = conv(v_1, \ldots, v_k)$ with $v_i \in \Re^n$ for all i. Prove that for any objective function $c \in \Re^n$, there is some v_j such that $c^T v_j \leq c^T x$ for all $x \in Q$.

4. Suppose that you are given a feasible solution $\bar{x}$ of value $\bar{\gamma}$ to the problem $\max(c^T x : Ax \leq b)$. Give a method that either demonstrates that the feasible region is unbounded (i.e., there is a point x and direction y such that $x + \lambda y$ is feasible for all $\lambda > 0$) or that finds a vertex $\tilde{x}$ of the feasible region with objective value $c\tilde{x} \geq \bar{\gamma}$. Your method should not use general purpose linear programming algorithms (like the simplex method). (Hint: some of the discussion of the equivalence of bounded polyhedra and polytopes, as well as the equivalence of extreme points, vertices, and basic feasible solutions, might prove useful.)