ORIE 6300 Mathematical Programming I November 14, 2008

Problem Set 10

Due Date: November 21, 2008

1. (21 points) Prove that the following problems are NP-complete.
   (a) (7 points) In the *partition* problem, we are given as input n numbers, $a_1, \ldots, a_n$. We must decide whether or not there is a partition of the indices $N = \{1, \ldots, n\}$ into two sets $S \subseteq N$ and $N - S$ such that $\sum_{i \in S} a_i = \sum_{i \in N-S} a_i$.
   (b) (7 points) A *dominating set* of an undirected graph G is a set of vertices S such that for all $u \notin S$, there is some edge (u, v) with $v \in S$. In the decision version of this problem, we are given the undirected graph and an input K and must decide if there exists a dominating set S such that $|S| \leq K$.
   (c) (7 points) In the decision version of the bin packing problem, we are given B bins of size 1 and n items of size $s_1, \ldots, s_n$, and must decide whether all the items can be packed into the B bins.

2. (30 points) In class, you saw a problem known as the vertex cover problem. In the *minimum-weight vertex cover problem*, you are given as input an undirected graph $G = (V, E)$ and weights $w_i \geq 0$ for all $i \in V$. The goal is to find a minimum-weight subset of vertices $S \subseteq V$ such that each edge $e \in E$ has at least one endpoint in S. The decision version of this problem is NP-complete. Consider the following integer programming formulation of the problem:

$$\begin{aligned} \text{Min} \sum_{i \in V} w_i x_i & & & \\ x_i + x_j & \geq & 1 & \quad \forall (i,j) \in E \\ x_i & \in & \{0,1\} & \quad \forall i \in V. \end{aligned}$$

   We can obtain a linear programming relaxation of the problem by replacing the constraints $x_i \in \{0,1\}$ with $x_i \geq 0$.

   (a) (10 points) Show that for any extreme point of the linear programming relaxation $x_i \in \{0, \frac{1}{2}, 1\}$ for all $i \in V$.
   (b) (10 points) Assuming we can solve linear programs in polynomial time, give a polynomial-time algorithm that finds a vertex cover of cost at most twice the cost of an optimal solution to the integer program.

   We can do slightly better than this on restricted classes of graphs. A *planar graph* is one that can be drawn in the plane using points for vertices and curves for edges such that no two curves intersect. An interesting fact is that the vertices of a planar graph can be colored with four colors such that each edge has two differently colored endpoints.

   (c) (10 points) Given a polynomial-time algorithm for computing a 4-coloring of a planar graph, show that you can find a polynomial-time algorithm for the minimum-weight vertex cover problem on planar graphs that returns a vertex cover of cost at most $\frac{3}{2}$ times the optimal solution to the integer program.