ORIE 6300 Mathematical Programming I September 4, 2008

Problem Set 1

Due Date: September 12, 2008

1. (a) Consider the LP:

$$\begin{array}{rcl} \text{Min } c^T x + \bar{c}^T \bar{x} & & \\ Ax + \bar{A}\bar{x} & = & b \\ x & \geq & 0 \\ \bar{x} & & \text{unconstrained} \end{array}$$

   Suppose we want to convert this problem to one in standard form; that is, with all variables being nonnegative. In class, we saw that one could do this by replacing the unconstrained variables with the difference of nonnegative variables (e.g. $\bar{x} = t - s$ for $s, t \geq 0$), but this doubles the number of such variables. Devise another technique to obtain an equivalent standard form problem where the number of variables is only increased by one.

   (b) Consider the LP:

$$\begin{array}{rcl} \text{Max } y^T b & & \\ A^T y & \leq & c \\ \bar{A}^T y & = & \bar{c} \\ y & & \text{unconstrained} \end{array}$$

   We want to convert this into a problem in the form of the dual to a standard form problem; i.e. with all less-than-or-equal-to constraints. The usual way to do this is to replace each equality constraint by two inequality constraints, but this doubles the number of such constraints. Devise another technique that only increase the number of constraints by one.

   (c) What is the relationship between the techniques in (a) and (b)?

2. What is the dual of the linear program with variables $x \geq 0$ and an additional single variable λ with constraints $Ax \leq \lambda b$, where the objective is to minimize λ?

3. In the general setting for the maximum-flow minimum-cut theorem, the input consists of a graph $G = (V, E)$ (with vertex set V and arc set E), source s, sink t, and a capacity u_e for each arc $e \in E$. For a cut (S, T), one typically defines its capacity to be

$$\sum_{e=uv: u \in S, v \in T} u_e.$$

   Consider the following linear programming formulation (using the notation from the lecture notes) of the maximum flow problem:

$$
\begin{aligned}
\max \sum_{P \in \mathcal{P}} x_P & & \\
x_P &\geq 0 \text{ for each path } P \in \mathcal{P} \\
\sum_{P : e \in P} x_P &\leq u_e \text{ for each edge } e \in E
\end{aligned}
$$

State and prove (using a generalization of the same probabilistic argument used in class) a "maximum-flow, minimum-cut" theorem generalizing the one given in class to allow for general capacities.