ORIE 6300 Mathematical Programming I November 20, 2008

Lecture 23

Lecturer: David P. Williamson *Scribe: Amy Cochran*

1 Termination of the Ellipsoid Method

1.1 Modifying the LP

Consider the system $Cx \leq d$. Let L be the number of bits used to encode C and d. Recall that, via Cramer's Rule, at a vertex x, $|x| \leq 2^L$. We can restate our system as follows:

$$\begin{aligned} Cx &\leq d \\ x &\leq 2^L e \\ -x &\leq 2^L e, \end{aligned}$$

where e is the vector of all ones.

Lemma 1 *1. If $Cx \leq d$ is infeasible, then $Cx \leq d + \frac{2^{-L}}{n+2}e$ is also infeasible.*

2. If $Cx \leq d$ is feasible, then $\exists \hat{x}$ such that

$$B\left(\hat{x}, \frac{2^{-2L}}{n+2}\right) \subseteq \{x : Cx \leq d + \frac{2^{-L}}{n+2}e\},$$

where $B(x,r)$ is a ball centered at x with radius r.

Idea: Run ellipsoid method on the system $Cx \leq d + \frac{2^{-L}}{n+2}e$, since $Cx \leq d$ feasible implies that the new system has volume greater than or equal to that of the $B(\hat{x}, \frac{2^{-2L}}{n+2})$. If $Cx \leq d$ is not feasible, then the new system is also infeasible.

Proof:

1. If $Cx \leq d$ is infeasible, then by Farkas' Lemma, we know that

$$c^T y = 0, d^T y = -1, y \geq 0$$

is feasible. As we claimed before, for any vertex solution, $\hat{y}$,

$$|\hat{y}| \leq 2^L e.$$

Also, at most $n+1$ components of $\hat{y}$ are nonzero. Therefore,

$$\begin{aligned} \left(d + \frac{2^{-L}}{n+2}e\right)^T \hat{y} &= d^T \hat{y} + \frac{2^{-L}}{n+2}\sum_i \hat{y}_i \\ &< d^T y + 1 \\ &= -1 + 1 = 0. \end{aligned}$$

Then $\hat{y}$ satisfies

$$\begin{aligned} \left(d + \frac{2^{-L}}{n+2}\right)^T \hat{y} &< 0, \\ c^T \hat{y} &= 0 \\ \hat{y} &\geq 0. \end{aligned}$$

Then by Farkas' lemma,

$$Cx \leq d + \frac{2^{-L}}{n+2} e$$

is infeasible, satisfying part (1) of the lemma.

2. Let $\hat{x}$ be feasible for

$$\begin{aligned} Cx &\leq d, \\ \hat{x} &\leq 2^L e, \\ -\hat{x} &\leq 2^L e. \end{aligned}$$

Pick any $x \in B(\hat{x}, \frac{2^{-L}}{n+2})$. Then for the jth constraint:

$$\begin{aligned} C_j x &= C_j \hat{x} + C_j (x - \hat{x}), \\ &\leq d_j + \|C_j\| \|x - \hat{x}\|, \\ &\leq d_j + 2^L \frac{2^{-2L}}{n+2}, \\ &= d_j + \frac{2^{-L}}{n+2}, \end{aligned}$$

where C_j is the jth row of the matrix C. Hence x is feasible for $Cx \leq d + \frac{2^{-L}}{n+2} e$. Since $x \in B(\hat{x}, \frac{2^{-2L}}{n+2})$ was arbitrary, so it must be that

$$B\left(\hat{x}, \frac{2^{-2L}}{n+2}\right) \subseteq \{x : Cx \leq d + \frac{2^{-L}}{n+2})\},$$

satisfying part (2) of the lemma.

□

Question: What happens if we get solution

$$\hat{x} \in \{x : Cx \leq d + \frac{2^{-L}}{n+2} e\}$$

but $C\hat{x} \not\leq d$?

Idea: Maintain a set, I, of tight constraints.

Consider the following algorithm to get the set I:

For $i \leftarrow 1$ to m

Check feasibility of

$$\begin{aligned} c_j x &= d_j \qquad \forall j \in I \\ c_i x &= d_i \\ c_j x &= d_j \qquad j = i+1, ..., m. \end{aligned}$$

If feasible

$$I \leftarrow I \cup \{i\}$$

Once the set I is formed then solve $C_i x = d_i$ for all $i \in I$ by Gaussian elimination.

So we see that if we add the bounding constraints and perturb the RHS, we obtain a system for which the ellipsoid method will terminate, and in the case when the obtained solution is infeasible in the original system, we have an algorithm to find a feasible solution.

1.2 Using Ellipsoid Method to Solve LPs

Previously, we gave a 3-step process for finding optimal solution to min $c^T x, Ax \leq b, x \geq 0$ using ellipsoid method. Instead,

- If current center a_k is not feasible since $A_i a_k > b_k$ for some i, then use hyperplane $A_i x \leq A_i a_k$ to break the region into two parts.
- If a_k is feasible, use hyperplane $c^T x \leq c^T a_k$ to break region into two parts and keep the region that contains an optimal solution. (This is called a objective function cut.)

This idea can be extended to a polynomial time algorithm for optimizing linear programs.

2 Application of the Ellipsoid Method

2.1 Applying the Method to TSP LP

Recall the linear programming relaxation of the traveling salesman problem:

$$\begin{aligned} \text{Min} \sum_{e \in E} c_e x_e & \\ \sum_{e \in \delta(s)} x_e &\geq 2, \qquad \forall S \subset V, S \neq \emptyset \\ \sum_{e \in \delta(v)} x_e &= 2, \qquad \forall v \in V \\ x_e &\leq 1 \qquad \forall e \in E \\ x_e &\geq 0 \qquad \forall e \in E. \end{aligned}$$

We want to solve this LP in polynomial time, but the number of constraints is exponential in $|V|$. To run the ellipsoid method, we need to determine if the center is feasible, and, in the case when it is infeasible, give a violated constraint. If we can do this in polynomial time, then we can solve the LP in polynomial time.

Question: For TSP LP, can we tell if x is feasible and find the violated constraint if it is not feasible?

Idea: If we treat x_e as a capacity we can apply the idea of minimum s-t cuts to the problem and use the min-cut/max-flow theorem.

Pick arbitrary $s, t \in V$. If $\exists S \subset V$ such that

$$\sum_{e \in \delta(S)} x_e < 2, \qquad s \in S, t \not\ni S,$$

then the maximum s-t flow is less than 2. Max-flow problems can be solved in polynomial time. Therefore, we check to see if

$$\sum_{e \in \delta(S)} x_e \geq 2$$

for all possible s-t pairs. If the flow values are at least 2 for all possible $s, t \in V$, then we know that all the constraints

$$\sum_{e \in \delta(S)} x_e \geq 2$$

are satisfied for all S. If the flow value is less than 2 for some s-t pair, then the corresponding minimum s-t cut gives a violated constraint of the linear program. Hence we can either detect if x is feasible or find a violated constraint if x is not feasible in polynomial time.

2.2 More General Problems

Given a polynomial time separation oracle, LPs can be solved in polynomial time (modulo some technicalities). More general problems can be solved using the ellipsoid method, e.g. convex regions, given an algorithm that finds separating hyperplanes.