1 Ellipsoid Method for LP (cont’d)

Recall that we want to build an ellipsoid method to decide if \( \exists x \in P = \{ x \in \mathbb{R}^n : Cx \leq d \} \), which returns \( x \in P \) if any exists, or output “infeasible” if \( P = \emptyset \). For convenience, we assume \( P \) is bounded. Recall that we defined \( L \) to be the number of bits needed to represent \( C, d \) in binary.

Let \( E_0 \) be a sphere centered at \( a_0 = 0 \), with radius \( 2^L \). It has been shown last time that we have \( P \subseteq E_0 \), and \( \text{volume}(E_0) = 2^{O(nL)} \).

Here is the ellipsoid method. In each step, if the center of the ellipsoid \( a_k \in P \), then the purpose has been achieved, return \( a_k \); otherwise, there exists \( C_j \), a row of \( C \), such that \( C_j a_k > d_j \). Compute a new ellipsoid

\[
E_{k+1} \supseteq E_k \cap \{ x : C_j x \leq C_j a_k \}.
\]

Repeat.

By construction, if \( P \subseteq E_k \), then \( P \subseteq E_{k+1} \). We claimed the following last time:

1. After \( O(n) \) iterations, the volume of current ellipsoid has dropped by a factor of at least 2.

2. If volume of \( E_k \) is \( 2^{-cnL} \) for some \( c \), and \( P \subseteq E_{k+1} \), then \( P = \emptyset \).

If these two claims are true, then after \( O(n^2 L) \) iterations, either the algorithm outputs some \( x \in P \), or it correctly stops, and outputs “infeasible”.

\[\text{Figure 1: General Case for Unit Sphere}\]

\[\text{Figure 2: The Case Solved in Problem Set}\]

From last time and the question in the problem set, we know if \( E_0 \) is the \( n \)-dimensional unit sphere (with \( a_0 = 0 \)), and \( C_j = -e_1^T \), then for

\[
E_1 = \left\{ x \in \mathbb{R}^n : \left( \frac{n+1}{n} \right)^2 \left( x_1 - \frac{1}{n+1} \right)^2 + \frac{n^2 - 1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\},
\]
then \( E_1 \supseteq E_0 \cap \{ x : c_j x \leq c_j a_0 \} \), and volume\((E_1) \leq e^{-\frac{1}{2(n+1)}} \text{volume}(E_0)\). Today, we want to extend this result to general case, where \( E = E_0 \) is any ellipsoid with center \( a = a_0 \), and \( C_j = c^T \) is any constraint. We will show that there exists \( E' \) and \( a' \), such that \( E' \supseteq E_0 \cap \{ x : c^T x \leq c^T a_0 \} \), and \( \text{volume}(E') \leq e^{-\frac{1}{2(n+1)}} \text{volume}(E) \). Note that this implies \( \text{volume}(E_{k+2(n+1)}) \leq e^{-1} \text{volume}(E_k) \), as claimed.

We will write an ellipsoid given its center \( a \) and a matrix \( A \) as:

\[
E(a, A) = \left\{ x \in \mathbb{R}^n : (x - a)^T A^{-1} (x - a) \leq 1 \right\},
\]

where the matrix \( A \) should be symmetric and positive definite (that is, \( v^T A v > 0, \forall v \in \mathbb{R}^n \)). Thus for the ellipsoid that we saw before, \( E_1 = E(a, A) \) for \( a = \frac{1}{n+1} e_1 \), \( A = \frac{n^2}{n^2-1} \left( I - \frac{2}{n+1} e_1 e^T_1 \right) \).

First, suppose that \( E_0 = E(0, I) \), the unit sphere centered at origin, but now we have arbitrary constraint \( c \). Assume \( ||c|| = 1 \). (i.e., \( c^T c = 1 \)). In order to handle this, the main idea is to reduce to previous case. Consider applying a rotation \( y = T(x) \), so that \( -e_1 = T(c) \). Then rotate \( E_1 \) back using \( T^{-1} \).

Since \( T \) is a rotation, \( y = T(x) = U x \) for some orthonormal matrix \( U \) \((U^T = U^{-1})\). We want \( Uc = -e_1 \), so \( c = -U^{-1} e_1 = -U^T e_1 \). In the transformed space, the desired ellipsoid is \( \{ x \in \mathbb{R}^n : (Ux - a)^T A^{-1} (Ux - a) \leq 1 \} \). Since \( U^T U = I \), this is the same as \( \{ x : (Ux - a)^T U U^T A^{-1} U U^T (Ux - a) \leq 1 \} \).

![Figure 3: Rotation](image_url)

Now we observe that

\[
(Ux - a)^T U = ((Ux)^T - a^T) U \nonumber = (x^T U^T - a^T) U \nonumber = x^T - a^T U \nonumber = (x - U^T a)^T, \nonumber
\]

and

\[
U^T (Ux - a) = x - U^T a, \nonumber
\]

where we define

\[
U^T a = U^T \left( \frac{1}{n+1} e_1 \right) = - \frac{1}{n+1} e =: \hat{a}. \nonumber
\]
If we set $\hat{A}^{-1} = U^T A^{-1} U$, then we get
\[
\hat{A} = (U^T A^{-1} U)^{-1} = U^{-1} A (U^{-1})^T
= \frac{n^2}{n^2 - 1} U^T (I - \frac{2}{n + 1} e_1 e_1^T) U
= \frac{n^2}{n^2 - 1} (I - \frac{2}{n + 1} (U^T e_1) (e_1^T U))
= \frac{n^2}{n^2 - 1} (I - \frac{2}{n + 1} (-c)(-c^T))
= \frac{n^2}{n^2 - 1} (I - \frac{2}{n + 1} cc^T).
\]

Therefore in this case,
\[
E' = \{ x \in \mathbb{R}^n : (x - \hat{a})^T \hat{A}^{-1} (x - \hat{a}) \leq 1 \}.
\]

Since we only performed a rotation, the volume did not change. So $\text{volume}(E') \leq e^{-\frac{1}{2n+1}} \text{volume}(E_0)$.

Now what if $E$ is not the unit sphere but a general ellipsoid? The idea is to transform $E$ into unit sphere centered at origin via transform $T(x) = y$, apply the result of the previous case, then transform it back via $T^{-1}$.

![Figure 4: Case of General Ellipsoid](image)

Let $E = E_k = E(a_k, A_k)$. Since $A_k$ is positive definite, $A_k = B^T B$ for some $B$. Then $A_k^{-1} = B^{-1} (B^{-1})^T$, and
\[
E(a_k, A_k) = \{ x : (x - a_k)^T B^{-1} (B^{-1})^T (x - a_k) \leq 1 \}.
\]

If we set $y = T(x) = (B^{-1})^T (x - a_k)$, we will get
\[
y^T y \leq 1.
\]

So $T$ transforms $E_k$ into $E(0, I)$. $T^{-1}(y) = x = B^T y + a_k$.  

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The hyperplane in the original space \( d^T x \leq d^T a_k \) becomes \( d^T (B^T y + a_k) \leq d^T a_k \), thus \( d^T B^T y \leq 0 \) after the transform \( T \). We want \( c^T y \leq 0 \) for \( \|c\| = 1 \), therefore set

\[
c^T = \frac{d^T B^T}{\|d^T B^T\|},
\]

hence

\[
c = \frac{Bd}{\sqrt{d^T Ad}}.
\]

In the transformed space, we have

\[
E' = \left\{ y : (y + \frac{1}{n+1} c)^T F^{-1} (y + \frac{1}{n+1} c) \leq 1 \right\},
\]

where

\[
F = \hat{A} = \frac{n^2}{n^2 - 1} (I - \frac{2}{n+1} cc^T).
\]

Now substitute \( y = (B^{-1})^T (x - a_k) \) to get back to the original space. We have

\[
E_{k+1} = \left\{ x : (B^{-1})^T (x - a_k) + \frac{1}{n+1} c)^T F^{-1} (B^{-1})^T (x - a_k) + \frac{1}{n+1} c \leq 1 \right\},
\]

\[
E_{k+1} = \left\{ x : (x - a_k)^T B^{-1} + \frac{1}{n+1} c^T \right\} F^{-1} \left( (B^{-1})^T (x - a_k) + \frac{1}{n+1} c \right) \leq 1 \right\}.
\]

If we set \( a_{k+1} = a_k - \frac{1}{n+1} B^T c \), then

\[
E_{k+1} = \left\{ x : (x - a_{k+1})^T B^{-1} F^{-1} (B^{-1})^T (x - a_{k+1}) \leq 1 \right\}.
\]

If we set \( \hat{F} = B^{-1} F^{-1} (B^{-1})^T \), then

\[
\hat{F} = B^T F B = \frac{n^2}{n^2 - 1} B^T \left( I - \frac{2}{n-1} cc^T \right) B
\]

\[
= \frac{n^2}{n^2 - 1} \left( A_k - \frac{2}{n+1} (B^T c)(B^T c)^T \right)
\]

\[
= \frac{n^2}{n^2 - 1} \left( A_k - \frac{2}{n+1} bb^T \right),
\]

where we set \( b = B^T c \). Then \( a_{k+1} = a_k - \frac{h}{n+1} \), and \( A_{k+1} = \hat{F} = \frac{n^2}{n^2 - 1} \left( A_k - \frac{2}{n+1} bb^T \right) \).

Since the ratios of volumes are preserved under linear transformation,

\[
\frac{\text{volume}(E_{k+1})}{\text{volume}(E_k)} = \frac{\text{volume}(E')}{\text{volume}(E_0)} \leq e^{-\frac{1}{(n+1)}}.
\]