ORIE 6300 Mathematical Programming I	David P. Williamson

Lecture 22

Lecturer: November 18, 2008 *Scribe: Yi Shen*

1 Ellipsoid Method for LP (cont'd)

Recall that we want to build an ellipsoid method to decide if $\exists x \in P = \{x \in \mathbb{R}^n : Cx \leq d\}$, which returns $x \in P$ if any exists, or output "infeasible" if $P = \emptyset$. For convenience, we assume P is bounded. Recall that we defined L to be the number of bits needed to represent C, d in binary.

Let E_0 be a sphere centered at $a_0 = 0$, with radius 2^L. It has been shown last time that we have $P \subseteq E_0$, and volume$(E_0) = 2^{O(nL)}$.

Here is the ellipsoid method. In each step, if the center of the ellipsoid $a_k \in P$, then the purpose has been achieved, return a_k; otherwise, there exists C_j, a row of C, such that $C_j a_k > d_j$. Compute a new ellipsoid

$$E_{k+1} \supseteq E_k \cap \{x : C_j x \leq C_j a_k\}.$$

Repeat.

By construction, if $P \subseteq E_k$, then $P \subseteq E_{k+1}$. We claimed the following last time:

1. After $O(n)$ iterations, the volume of current ellipsoid has dropped by a factor of at least 2.
2. If volume of E_k is 2^{-cnL} for some c, and $P \subseteq E_{k+1}$, then $P = \emptyset$.

If these two claims are true, then after $O(n^2 L)$ iterations, either the algorithm outputs some $x \in P$, or it correctly stops, and outputs "infeasible".

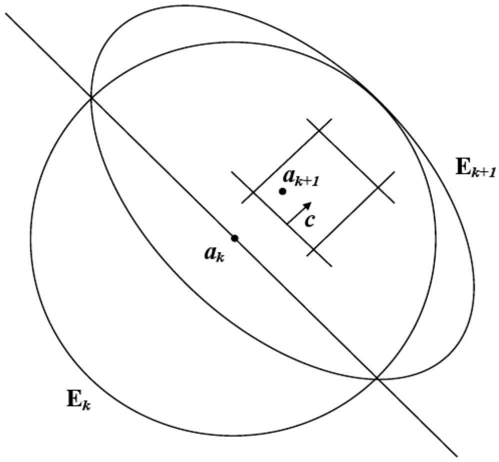

Figure 1: *General Case for Unit Sphere*

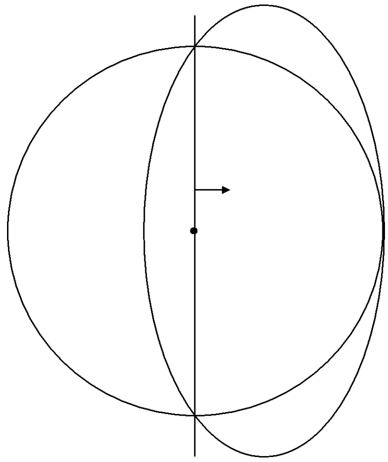

Figure 2: *The Case Solved in Problem Set*

From last time and the question in the problem set, we know if E_0 is the n-dimensional unit sphere (with $a_0 = 0$), and $C_j = -e_1^T$, then for

$$E_1 = \left\{ x \in \mathbb{R}^n : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\},$$

then $E_1 \supseteq E_0 \cap \{x : c_j x \leq c_j a_0\}$, and volume$(E_1) \leq e^{-\frac{1}{2(n+1)}}$volume$(E_0)$. Today, we want to extend this result to general case, where $E = E_0$ is any ellipsoid with center $a = a_0$, and $C_j = c^T$ is any constraint. We will show that there exists E' and a', such that $E' \supseteq E \cap \{x : c^T x \leq c^T a\}$, and volume$(E') \leq e^{-\frac{1}{2(n+1)}}$volume$(E)$. Note that this implies volume$(E_{k+2(n+1)}) \leq e^{-1}$volume$(E_k)$, as claimed.

We will write an ellipsoid given its center a and a matrix A as:

$$E(a, A) = \{x \in \mathbb{R}^n : (x-a)^T A^{-1}(x-a) \leq 1\},$$

where the matrix A should be symmetric and positive definite (that is, $v^T A v > 0,\ \forall v \in \mathbb{R}^n$). Thus for the ellipsoid that we saw before, $E_1 = E(a, A)$ for $a = \frac{1}{n+1}e_1$, $A = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1}e_1 e_1^T\right)$.

First, suppose that $E_0 = E(0, I)$, the unit sphere centered at origin, but now we have arbitrary constraint c. Assume $\|c\| = 1$. (*i.e.*, $c^T c = 1$). In order to handle this, the main idea is to reduce to previous case. Consider applying a rotation $y = T(x)$, so that $-e_1 = T(c)$. Then rotate E_1 back using T^{-1}.

Since T is a rotation, $y = T(x) = Ux$ for some orthonormal matrix U ($U^T = U^{-1}$). We want $Uc = -e_1$, so $c = -U^{-1}e_1 = -U^T e_1$. In the transformed space, the desired ellipsoid is $\{x \in \mathbb{R}^n : (Ux-a)^T A^{-1}(Ux-a) \leq 1\}$. Since $U^T U = I$, this is the same as $\{x : (Ux - a)^T U U^T A^{-1} U U^T (Ux-a) \leq 1\}$.

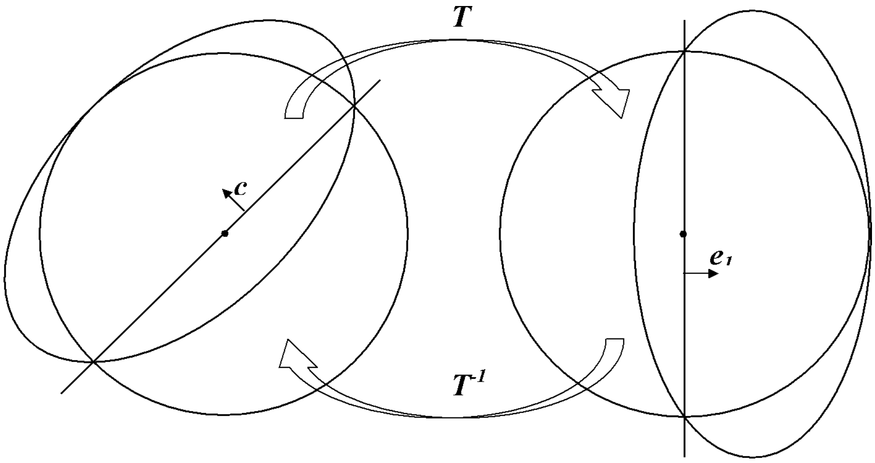

Figure 3: *Rotation*

Now we observe that

$$\begin{aligned}(Ux-a)^T U &= ((Ux)^T - a^T)U \\ &= (x^T U^T - a^T)U \\ &= x^T - a^T U \\ &= (x - U^T a)^T,\end{aligned}$$

and

$$U^T(Ux - a) = x - U^T a,$$

where we define

$$U^T a = U^T\left(\frac{1}{n+1}e_1\right) = -\frac{1}{n+1}e =: \hat{a}.$$

If we set $\hat{A}^{-1} = U^T A^{-1} U$, then we get

$$\begin{aligned}
\hat{A} &= (U^T A^{-1} U)^{-1} \\
&= U^{-1} A (U^{-1})^T \\
&= \frac{n^2}{n^2-1} U^T (I - \frac{2}{n+1} e_1 e_1^T) U \\
&= \frac{n^2}{n^2-1} (I - \frac{2}{n+1} (U^T e_1)(e_1^T U)) \\
&= \frac{n^2}{n^2-1} (I - \frac{2}{n+1} (-c)(-c^T)) \\
&= \frac{n^2}{n^2-1} (I - \frac{2}{n+1} c c^T).
\end{aligned}$$

Therefore in this case,

$$E' = \{x \in \mathbb{R}^n : (x - \hat{a})^T \hat{A}^{-1} (x - \hat{a}) \leq 1\}.$$

Since we only performed a rotation, the volume did not change. So volume$(E') \leq e^{-\frac{1}{2(n+1)}}$ volume(E_0).

Now what if E is not the unit sphere but a general ellipsoid? The idea is to transform E into unit sphere centered at origin via transform $T(x) = y$, apply the result of the previous case, then transform it back via T^{-1}.

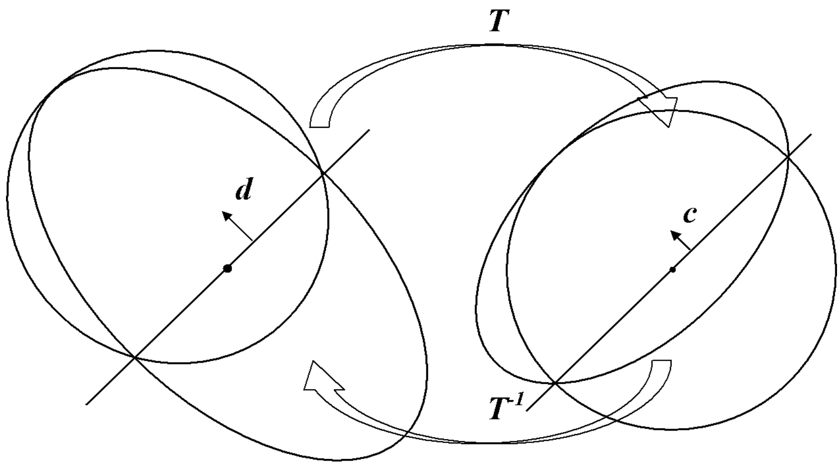

Figure 4: *Case of General Ellipsoid*

Let $E = E_k = E(a_k, A_k)$. Since A_k is positive definite, $A_k = B^T B$ for some B. Then $A_k^{-1} = B^{-1}(B^{-1})^T$, and

$$E(a_k, A_k) = \{x : (x - a_k)^T B^{-1} (B^{-1})^T (x - a_k) \leq 1\}.$$

If we set $y = T(x) = (B^{-1})^T (x - a_k)$, we will get

$$y^T y \leq 1.$$

So T transforms E_k into $E(0, I)$. $T^{-1}(y) = x = B^T y + a_k$.

The hyperplane in the original space $d^T x \le d^T a_k$ becomes $d^T(B^T y + a_k) \le d^T a_k$, thus $d^T B^T y \le 0$ after the transform T. We want $c^T y \le 0$ for $\|c\| = 1$, therefore set

$$c^T = \frac{d^T B^T}{\|d^T B^T\|},$$

hence

$$c = \frac{Bd}{\sqrt{d^T A d}}.$$

In the transformed space, we have

$$E' = \left\{ y : (y + \frac{1}{n+1} c)^T F^{-1} (y + \frac{1}{n+1} c) \le 1 \right\},$$

where

$$F = \hat{A} = \frac{n^2}{n^2 - 1}(I - \frac{2}{n+1} c c^T).$$

Now substitute $y = (B^{-1})^T (x - a_k)$ to get back to the original space. We have

$$E_{k+1} = \left\{ x : \left((B^{-1})^T (x - a_k) + \frac{1}{n+1} c \right)^T F^{-1} \left((B^{-1})^T (x - a_k) + \frac{1}{n+1} c \right) \le 1 \right\},$$

$$E_{k+1} = \left\{ x : \left((x - a_k)^T B^{-1} + \frac{1}{n+1} c^T \right) F^{-1} \left((B^{-1})^T (x - a_k) + \frac{1}{n+1} c \right) \le 1 \right\}.$$

If we set $a_{k+1} = a_k - \frac{1}{n+1} B^T c$, then

$$E_{k+1} = \{ x : (x - a_{k+1})^T B^{-1} F^{-1} (B^{-1})^T (x - a_{k+1}) \le 1 \}.$$

If we set $\hat{F}^{-1} = B^{-1} F^{-1} (B^{-1})^T$, then

$$\begin{aligned}
\hat{F} = B^T F B &= \frac{n^2}{n^2 - 1} B^T \left(I - \frac{2}{n-1} c c^T \right) B \\
&= \frac{n^2}{n^2 - 1} \left(A_k - \frac{2}{n+1} (B^T c)(B^T c)^T \right) \\
&= \frac{n^2}{n^2 - 1} \left(A_k - \frac{2}{n+1} b b^T \right),
\end{aligned}$$

where we set $b = B^T c$. Then $a_{k+1} = a_k - \frac{b}{n+1}$, and $A_{k+1} = \hat{F} = \frac{n^2}{n^2-1} \left(A_k - \frac{2}{n+1} b b^T \right)$.

Since the ratios of volumes are preserved under linear transformation,

$$\frac{\text{volume}(E_{k+1})}{\text{volume}(E_k)} = \frac{\text{volume}(E')}{\text{volume}(E_0)} \le e^{-\frac{1}{2(n+1)}}.$$