ORIE 6300 Mathematical Programming I	October 23, 2008
Lecture 15	
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1 Varieties of Simplex Method: Dual Simplex

1.1 Description

Recall that the regular (primal) simplex method is an algorithm that maintains primal feasibility and works towards dual feasibility. We start with a primal feasible solution and try to reach dual feasibility while maintaining complementary slackness. Dual simplex is exactly analogous to primal simplex where we start with a dual feasible solution corresponding to a basis B and move towards making the corresponding primal solution feasible while maintaining complementary slackness.

Consider the standard primal and dual linear programs.

$$\begin{array}{llcll} \min & c^T x & \qquad & \max & b^T y \\ \text{s.t.} & Ax = b & & \text{s.t.} & A^T y \le c \\ & x \ge 0 & & & \end{array}$$

Assume we have a dual basic feasible solution $y = (A_B^T)^{-1} c_B$ with associated basis B, then

$$A^T y \le c \quad \text{i.e.} \quad \bar{c} \ge 0$$

Let

$$x_B = A_B^{-1} b = \bar{b}, \quad x_N = 0$$

If $\bar{b} \ge 0$, then x is primal feasible. Since x and y satisfy complementary slackness, they are primal and dual optimal solutions. If not, then there exists $i \in B$ such that $\bar{b}_i < 0$. So we want to remove i from the basis B. The next question is which index should we add to the basis.

Recall that the primal LP can be rewritten as

$$\begin{array}{cllccll} & \min & c_B^T x_B + c_N^T x_N & & & \min & c_B^T x_B + c_N^T x_N \\ (1) & \text{s.t.} & A_B x_B + A_N x_N = b & \text{or} & (2) & \text{s.t.} & I x_B + \bar{A} x_N = \bar{b} \\ & & x_B, x_N \ge 0 & & & & x_B, x_N \ge 0 \end{array}$$

where $\bar{A} = A_B^{-1} A_N$, $\bar{b} = A_B^{-1} b$. Consider the i^{th} constraint of LP(2),

$$x_i + \sum_{j \in N} \bar{A}_{ij} x_j = \bar{b}_i < 0$$

If $\bar{A}_{ij} \ge 0, \forall j \in N$, since we also have $x \ge 0$, thus above constraint cannot be satisfied. Therefore there is no feasible solution to the primal LP in this case.

Now what should we do if there exists $j \in N$ such that $\bar{A}_{ij} < 0$? Taking the duals for both (1) and (2), we get

$$\begin{array}{cllccll} & \max & b^T y & & & \max & \bar{b}^T \tilde{y} \\ (3) & \text{s.t.} & A_B^T y \le c_B & \text{or} & (4) & \text{s.t.} & I^T \tilde{y} \le c_B \\ & & A_N^T y \le c_N & & & & \bar{A}^T \tilde{y} \le c_N \end{array}$$

Note (3) and (4) are equivalent if we let $\tilde{y} = A_B^T y$. And we have already set y such that $A_B^T y = c_B$, or equivalently $\tilde{y} = c_B$. Consider LP (4). since $\bar{b}_i < 0$, we can increase the value of the objective function if we decrease $\tilde{y}_i$. But how far can we do this?

Suppose we decrease $\tilde{y}_i$ by δ. For any $j \in N$ such that $\bar{A}_{ij} \geq 0$, we still have $\bar{A}_j^T \tilde{y} \leq c_j$. For any $j \in N$ such that $\bar{A}_{ij} < 0$, the LHS of the j^{th} constraint goes up by $-\bar{A}_{ij}\delta$. To stay feasible, we should have

$$\delta \leq \frac{c_j - \bar{A}_j^T \tilde{y}}{-\bar{A}_{ij}} \quad \forall j \in N \text{ s.t. } -\bar{A}_{ij} < 0$$

We can rewrite it into a more familiar form

$$\delta \leq \frac{c_j - \bar{A}_j^T \tilde{y}}{-\bar{A}_{ij}} = \frac{c_j - (A_B^{-1} A_j)^T \tilde{y}}{-\bar{A}_{ij}} = \frac{c_j - A_j^T (A_B^T)^{-1} \tilde{y}}{-\bar{A}_{ij}} = \frac{c_j - A_j^T y}{-\bar{A}_{ij}} = \frac{\bar{c}_j}{-\bar{A}_{ij}}$$

Therefore if we decrease $\tilde{y}_i$ by δ such that

$$\delta = \min_{j \in N : \bar{A}_{ij} < 0} \frac{\bar{c}_j}{-\bar{A}_{ij}}$$

then the dual variables are still feasible. And the index j that achieves this minimum will enter the basis.

1.2 Summary

In the dual simplex method,

- Set $x = A_B^{-1} b$, $y = (A_B^T)^{-1} c_B$. If $\bar{b} \geq 0$, then x and y are primal and dual optimal.
- (**Check for infeasibility**) Otherwise, $\bar{b}_i < 0$ for some i. If $\bar{A}_{ij} \geq 0$ for all $j \in N$, then the LP is primal infeasible.
- (**Ratio test**) Otherwise, $\bar{A}_{ij} < 0$ for some j. Compute

  $$\delta = \min_{j \in N : \bar{A}_{ij} < 0} \frac{\bar{c}_j}{-\bar{A}_{ij}}$$

  Pick $j \in N$ that attains the minimum as the index to add to the basis.
- (**Update basis**) $\hat{B} \leftarrow B - \{i\} \cup \{j\}$

1.3 Why use dual simplex?

- The dual simplex works better in practice.
  - It is usually easier to find initial dual feasible solutions.
    Since in practise we usually have $c \geq 0$, then $y = 0$ is a dual feasible solution.
  - The dual LP is often less degenerate.
- "Warm start"s: to solve another related LP after solving the first one.
  - If the objective function c changes, we use primal simplex with the previous primal optimum as an initial primal feasible solution.

- If the RHS of the constraints b changes, we use dual simplex with the previous dual optimum as an initial dual feasible solution.
- If an additional constraint is added to the primal LP, we use dual simplex.
  Now the dual LP has one more variable, if we set the new variable to be 0 and all other variables to be the previous dual optimum, we get a dual feasible solution for the new LP and can carry out dual simplex.

  This is frequently used in solving integer programming.

2 Sensitivity Analysis

In sensitivity analysis, we ask the question: How do solutions x and y change as input data (A, b, c) changes? We'll look at small, local changes of each of these in turn.

2.1 Changes in b

Suppose we increase b_i by δ. $b \to b+\delta e_i$ (e_i is a vector of 0s with 1 in i^{th} place.) Then $y = (A_B^T)^{-1}c_B$ stays the same and feasible. Let $x_N = 0$, $x_B = A_B^{-1}(b+\delta e_i)$, then complimentary slackness is still obeyed. If we also have $x_B \geq 0$, then x is feasible. Thus x and y are optimal solutions.

How does optimal objective funtion value change?

$$\triangle c = c_B^T(\triangle x_B) = \delta(c_B^T A_B^{-1} e_i) = \delta((A_B^T)^{-1}c_B)^T e_i = \delta y^T e_i = \delta y_i$$

So the optimal dual variable y_i gives the change in cost as we perturb the RHS b_i.
y_i is ***shadow price/marginal cost*** of b_i.

Now suppose we change b to $b \to b + \delta\hat{b}$. Then $y = (A_B^T)^{-1}c_B$ stays feasible. Let $x_N = 0$, $x_B = A_B^{-1}(b+\delta\hat{b})$. Then x and y stay optimal if $x_B \geq 0$. The objective function changes by

$$\triangle c = c_B^T(\triangle x_B) = \delta(c_B^T A_B^{-1}\hat{b}) = \delta((A_B^T)^{-1}c_B)^T\hat{b} = \delta y^T\hat{b}.$$

Next time we'll look at changes in the other input data.