ORIE 6300 Mathematical Programming I	October 21, 2008

Lecture 14

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1 Simplex Issues: Number of Pivots

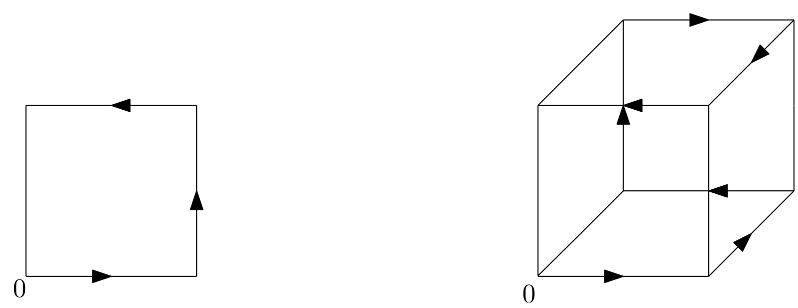
Question: How many pivots does the simplex algorithm need to take to find an optimal solution?
Answer: If we use a pivot rule that prevents cycling, like Bland's rule, we know that the simplex algorithm never revisits a basis. Therefore the number of pivots is bounded by the number of basis, which is no larger than $\binom{n}{m}$. In practice, however the simplex algorithm seems to perform $O(m)$ pivots.

For most known pivot rules, there have been examples devised where $2^n - 1$ pivots are taken. This work was started by Klee and Minty (1970).
Idea: Consider n-dimensional cube

$$0 \leq x_i \leq 1, \forall i = 1, 2, \ldots, n$$

There is a *Hamiltonian path* in any n-dimensional cube. This path starts at $x = 0$ and visits every vertex exactly once.

Examples:



The idea now is to perturb the costs and the constraints slightly so that the simplex algorithm follows the Hamiltonian path. Thus it performs $2^n - 1$ pivots. Observe, however that in such a case, only one pivot is necessary to reach the optimal solution!!!

Suppose now that P is a bounded polyhedron and we have two vertices of P, x and y. Let $d(x, y)$ be the least number of nondegenerate pivots needed to go from x to y. Let

$$D(P) = \max_{\textit{vertices: } x,y} d(x, y)$$

be the diameter of P. Let

$$\Delta(n, m) = \max_{P \in S} D(P),$$

where S is the set of all bounded polyhedra in $\mathbb{R}^n$ with m constraints. The following has been conjectured but is still an open question.

Theorem 1 *(Hirsch Conjecture):* $\Delta(n, m) \le m - n$

There have been some partial results in answering this question. For instance, the best upper bound known is $\Delta(n, m) \le m^{(1+\log n)}$.

2 Types of Simplex Method

2.1 Revised Simplex Method

This is an implementation of the simplex algorithm in which we maintain a basis B and compute all other information from it during each iteration. What we have called the simplex method throughout the semester thus far.

2.2 Standard Simplex Method

In this implementation of the simplex algorithm, we maintain a full tableau throughout the algorithm.

Tableau:

$-b(A_B^T)^{-1}c_B$	$c - A^T(A_B^T)^{-1}c_B$
$A_B^{-1}b$	$A_B^{-1}A$

- The top left hand corner has the negative value of the objective function. Recall that $\bar{c}x = c^Tx - b^Ty = c^Tx - b^T(A_B^T)^{-1}c_B$. But $\bar{c}x = 0$, since $\bar{c}_B = 0$ and $x_N = 0$. Therefore $c^Tx = b^T(A_B^T)^{-1}c_B$.
- The bottom left hand corner has the values of the basic variables.
- The top right hand corner has the reduced cost vector.
- The bottom right hand corner has a modified constrant matrix with the property that the submatrix consisting of the columns indexed by B is the identity I

Example of the standard Simplex Method

$$
\begin{array}{llllllllll}
\min & -x_1 & +2x_2 & -x_3 & & & & & \\
\text{s.t.} & x_1 & & & +\,x_4 & & & & = 4 \\
 & & x_2 & & & +\,x_5 & & & = 4 \\
 & x_1 & +x_2 & & & & +\,x_6 & & = 6 \\
 & -x_1 & & +\,2x_3 & & & & +\,x_7 & = 4
\end{array}
$$

Throughout there is the additional constraint that all variables be nonnegative.

0	-1	2	-1	0	0	0	0
x_4=4	1	0	0	1	0	0	0
x_5=4	0	1	0	0	1	0	0
x_6=6	1	1	0	0	0	1	0
x_7=4	-1	0	2	0	0	0	1

During this iteration, x_1 enters the basis and x_4 leaves the basis. To do an update take row defining the pivot, do Gaussian elimination to make the new basis submatrix into the Identity matrix. Also use this row to eliminate x_1 from the reduced costs.

4	0	2	-1	1	0	0	0
x_1=4	1	0	0	1	0	0	0
x_5=4	0	1	0	0	1	0	0
x_6=2	0	1	0	-1	0	1	0
x_7=8	0	0	2	1	0	0	1

During this iteration, x_3 enters the basis and x_7 leaves.

8	0	2	0	1.5	0	0	0.5
x_1=4	1	0	0	1	0	0	0
x_5=4	0	1	0	0	1	0	0
x_6=2	0	1	0	-1	0	1	0
x_3=4	0	0	1	0.5	0	0	0.5

We found the optimal solution, since the reduced costs vector has nonnegative entries. The value of the LP is -8 and the optimal solution is [4,0,4,0,4,2,0].

2.3 Capacitated Simplex

We consider an LP of the following form.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & l \leq x \leq u \end{aligned}$$

Question: How do we solve such an LP?
Idea 1: Put into standard form. Let $z = x - l$ and $\tilde{b} = b - Al$. The new LP is

$$\begin{aligned} \min \quad & c^T z \\ \text{s.t.} \quad & Az = \tilde{b} \\ & 0 \leq z \leq u - l \end{aligned}$$

Now we can convert $z \leq u - l$ to equality constraints by adding slack variables. Thus the number of costraints in the constraint matrix and the number of variables are increased by n. Since we typically have that $n >> m$, this alteration could cause the complexity of the algorithm to increase significantly.
Idea 2: Modify the Simplex Method! We will maintain three sets of variables: B (basic), L, and U. Then

$$j \in L \Longrightarrow x_j = l_j$$

$$j \in U \Longrightarrow x_j = u_j$$

If we take the dual of the original LP, we get

$$\begin{array}{rl} \max & b^T y - u^T v + l^T w \\ \text{s.t.} & A^T y - v + w = c \\ & v, w \geq 0 \end{array}$$

Notice that for any y, there exists a dual feasible solution; namely

$$v = \max(0, A^T y - c)$$

$$w = \max(0, c - A^T y)$$

We can set $y = (A_B^T)^{-1} c_B$ as usual and compute reduced costs, $\bar{c} = c - A^T y$. Let's now consider a solution to the primal, x.

$$A_B x_B + A_L x_L + A_U x_U = b$$

$$A_B x_B = b - A_L x_L - A_U x_U$$

$$x_B = A_B^{-1} b - A_B^{-1} A_L x_L - A_B^{-1} A_U x_U$$

If x satisfies the above equation and $l_B \leq x_B \leq u_B$, then it is primal feasible. Furthermore, it is optimal if it is primal feasible and $\bar{c_j} \geq 0$ for all $j \in L$, and $\bar{c_j} \leq 0$ for all $j \in U$. This follows since if $\bar{c_j} \geq 0$ then $A_j^T y \leq c_j \Longrightarrow v_j = 0$ and $w_j \geq 0$. Therefore complementary slackness is obeyed since $w_j > 0 \Longrightarrow x_j = l_j$. If $\bar{c_j} \leq 0$ then $A_j^T y \geq c_j \Longrightarrow w_j = 0$ and $v_j \geq 0$. Therefore complementary slackness is obeyed since $v_j > 0 \Longrightarrow x_j = l_j$.

The rest of the details of the capacitated simplex method will be given as a homework problem.