ORIE 6300 Mathematical Programming I October 7, 2008

Lecture 11

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1 Simplex Method to Solve LPs

Recall the standard form of primal and dual LPs:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \max & y^T b \\ \text{s.t.} & A^T y \leq c \end{array}$$

Given a basic feasible solution x, and a basis B, we compute $y = (A_B^T)^{-1} c_B$, reduced costs $\bar{c} = c - A^T y$, and observe that then $\bar{c}_B = c_B - A_B^T (A_B^T)^{-1} c_B = 0$. Recall that $x_N = 0$ and $x_B = A_B^{-1} b$.

As we've discussed, if $\bar{c} \geq 0$, then x is optimal. Otherwise, consider $\bar{b} = A_B^{-1} b$, and $\bar{A} = A_B^{-1} A_N$. Then we showed that the primal problem corresponds to the linear program $\min \bar{c} + N^T x_N : \bar{A} x_N \leq \bar{b}, x_N \geq 0$. We pick $j \in N$ such that $\bar{c}_j < 0$.

Check for Unboundedness If $\bar{A}_{ij} \leq 0$ for all i, then the LP is unbounded. Otherwise, continue to the next step.

Ratio Test: Compute $\epsilon := \min_{i:\bar{A}_{ij}>0} \frac{\bar{b}_i}{\bar{A}_{ij}}$. Fix $i^* \in B$ as the row corresponding to the first constraint becoming tight as we increase the value of variable j. We construct a new solution $\hat{x}_j \leftarrow \epsilon$, $\hat{x}_k \leftarrow 0$ for all $k \in N - \{j\}$, and $\hat{x}_B = \bar{b} - \bar{A} \hat{x}_N$.

Update Basis: Create a new basis $\hat{B} = B - \{i^*\} \cup \{j\}$.

2 Example

$$\begin{array}{lrrrl} \min & -x_1 & +\ 2x_2 & -x_3 & \\ \text{s.t.} & x_1 & & & \leq 4 \\ & & x_2 & & \leq 4 \\ & x_1 & +\ x_2 & & \leq 6 \\ & -x_1 & & +\ 2x_3 & \leq 4 \end{array}$$

Throughout there is the additional constraint that all variables be nonnegative. We convert the above problem to standard form, introducing slack variables. Notice that these slack variables form an identity matrix which corresponds to a basis.

$$\begin{array}{lrrrrrrrl} \min & -x_1 & +2x_2 & -x_3 & & & & & \\ \text{s.t.} & x_1 & & & +\ x_4 & & & & = 4 \\ & & x_2 & & & +\ x_5 & & & = 4 \\ & x_1 & +x_2 & & & & +\ x_6 & & = 6 \\ & -x_1 & & +\ 2x_3 & & & & +\ x_7 & = 4 \end{array}$$

In matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \\ 4 \end{bmatrix}$$

The first three columns of the coefficient matrix refer to N and the last four columns refer to B.

Initialization of Simplex: $x_N = 0$ and $x_B = A_B^{-1} b = b = \begin{bmatrix} 4 & 4 & 6 & 4 \end{bmatrix}^T$
Solve $A_B^T y = c_B$ with $c_B = 0$. Thus $y = c_B A_B^{-1} = 0$ and $\bar{c} = c - yA = c$.
$\bar{c}$ is not strictly positive. Intuitively this makes sense because we can increase either x_1 or x_3 to decrease the objective function since we still have slack in all of the constraints.

Iteration 1 of Simplex: Choose a variable j corresponding to a negative reduced cost to enter the basis. Let's take $j = x_1$ and increase it until one of our constraints holds with equality. We can increase x_1 by $\min_{i:\bar{A}_{ij}>0} \frac{\bar{b}_i}{\bar{A}_{ij}} = 4$. This corresponds to constraint 1 and thus $i^* = 1$. The variable which leaves the basis is $j^* = x_4$.

Now do simple row operations to get a unit vector in the first column.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 8 \end{bmatrix}$$

Columns 2, 3, and 4 now refer to N and columns 1, 5, 6, and 7 now refer to B. Our new basic feasible solution is: $\begin{bmatrix} 4 & 0 & 0 & 0 & 4 & 2 & 8 \end{bmatrix}^T$.

We need to have $\bar{c}_B = 0$. Recall our old objective function: $-x_1 + 2x_2 - x_3$. Since x_1 is in the basis, the corresponding coefficient needs to be zero. Substitute $x_1 = 4 - x_4$ (which we get from the first constraint) into the objective function. Note that we will always be able to express the new entering basic variable in terms of strictly non-basic variables. From this we get our new (reduced) objective function: $-4 + 2x_2 - x_3 + x_4$, or $\bar{c} = \begin{bmatrix} 0 & 2 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$ which is equivalent to solving $yA_b = c_B$.

Iteration 2 of Simplex: Choose a variable j corresponding to a negative reduced cost to enter the basis. The only negative reduced cost corresponds to $j = x_3$. Let's increase x_3 until one of our constraints holds with equality. We can increase x_3 by $\min_{i:\bar{A}_{ij}>0} \frac{\bar{b}_i}{\bar{A}_{ij}} = 4$. Our limiting constraint is constraint 4 and thus $i^* = 4$. The variable which leaves the basis is $j^* = x_8$.

Now do simple row operations to get a unit vector in the third column. In this case, all we have to do is divide the last row by 2.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 4 \end{bmatrix}$$

Columns 2, 4, and 7 now refer to N and columns 1, 3, 5, and 6 now refer to B. Our new basic feasible solution is: $\begin{bmatrix} 4 & 0 & 4 & 0 & 4 & 2 & 0 \end{bmatrix}^T$.

Recall our old objective function: $-4 + 2x_2 - x_3 + x_4$. Since x_3 is now in the basis, the corresponding coefficient needs to be zero. Substitute $x_3 = 4 - 1/2x_4 - 1/2x_7$ (which we get from the fourth constraint) into the objective function. From this we get our new (reduced) objective function: $-8 + 2x_2 - 3/2x_4 + 1/2x_7$, or $\bar{c} = \begin{bmatrix} 0 & 2 & 0 & 3/2 & 0 & 0 & 1/2 \end{bmatrix}$.

Since $\bar{c} \geq 0$ we are finished with simplex; optimal solution x^* is $\begin{bmatrix} 4 & 0 & 4 & 0 & 4 & 2 & 0 \end{bmatrix}^T$.

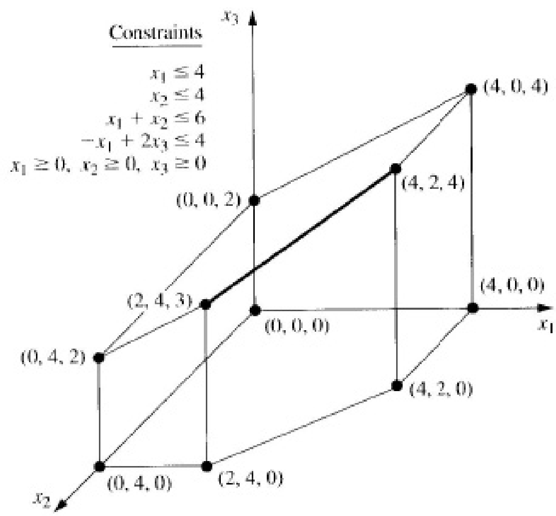

Figure 1: 3-D Diagram of the Feasible Region

The diagram above shows the simplex pivots graphically. We started at $x = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and then moved to $x = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}$ and finally to $x = \begin{bmatrix} 4 & 0 & 4 \end{bmatrix}$.

3 Finding an Initial Basic Feasible Solution

We now start addressing various issues with the simplex method. To start the Simplex algorithm, we need an initial basic feasible solution. The idea for achieving this is to modify the LP such that there is an easy initial basic feasible solution, and such that solving the modified LP gives some (not necessarily optimal) basic feasible solution to the original LP.

Without loss of generality we can assume $b \geq 0$; if $b_j < 0$ for some j, we multiply that constraint by -1 (this is okay since there are only equality constraints). Consider the problem

$$\begin{aligned} \min \quad & e^T z \\ \text{s.t.} \quad & Ax + Iz = b \\ & x \geq 0 \\ & z \geq 0, \end{aligned}$$

where $e = [1\ 1 \cdots 1]^T$ is a column of ones. The z variables are called *artificial variables*, and the x's are called *real variables*. Define $x' := [x^T\ z^T]^T$ and $A' := [A\ I]$ so that the constraints of the modified LP can be written as $A'x' = b$, $x' \geq 0$.

Let B be the indices corresponding to the artificial variables. Then B is a basis, since the corresponding columns of A' are I, the identity, and thus linearly independent. The corresponding basic feasible solution is $x = 0$, $z = b$. We use this to initialize the simplex algorithm.

The simplex method can one of two possible results (note that the modified LP is never unbounded: since $z \geq 0$, the objective function is bounded from below by 0.)

Case (1): The value of the LP is non-zero (and thus strictly greater than zero). Then there are no feasible solutions for the original LP, i.e., there are no x such that $Ax = b$. Indeed, if there were, we could take $z = 0$ and thus obtain a new feasible solution to the modified LP with value 0, a contradiction.

Case (2): The value of the LP is zero. Then there are two subcases:

(i) The Good Case: All artificial variables are non-basic. Then $A'_B = A_B$, so that B is a basis also for the original problem: $x'_B = (A'_B)^{-1}b$, $x'_N = 0$ is feasible, so $x_B = A_B^{-1}b$, $x_N = 0$ is a basic feasible solution. for $Ax = b$.

We can now run the simplex method for the original problem, starting with the basis B.

(ii) The Bad Case: Some artificial variables are in the basis.

We'll deal with the bad case next time.