ORIE 6300 Mathematical Programming I	October 2, 2008
Lecture 10	
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1 Recap of last class

Last time, we introduced the simplex method. In this class we are going to prove that the simplex method indeed works.

Let us first recall what the simplex method does. Consider the standard primal and its dual linear programs:

$$\begin{array}{llcll} \min & c^T x & \qquad & \max & y^T b \\ \text{s.t.} & Ax = b & & \text{s.t.} & A^T y \le c. \\ & x \ge 0 & & & \end{array}$$

We assume that we already have a starting point for the simplex method (we will address the issue of finding an initial basic feasible solution later on). Suppose we have a basic feasible solution x with associated basis B. We then have $x_N = 0$ and $x_B = A_B^{-1} b \ge 0$. Now we consider

$$y = (A_B^T)^{-1} c_B.$$

We showed last time that if y is dual feasible (i.e. $A^T y \le c$) then x is an optimal solution to the primal. In the case that y is not dual feasible we introduced the concept of reduced cost:

Definition 1 *For any $y \in \mathbb{R}^m$, the* **reduced cost** *$\bar{c}$, with respect to y, is $\bar{c} = c - A^T y$.*

Observation 1 *The reduced cost $\bar{c} \ge 0$, with respect to y, iff y is dual feasible.*

Lemma 1 *Consider the LP [min $c^T x$ s.t. $Ax = b$, $x \ge 0$] and the alternative LP [min $\bar{c}^T x$ s.t. $Ax = b$, $x \ge 0$]. Then $\hat{x}$ is an optimal solution for one iff it is optimal for the other.*

We showed that the primal LP can be rewritten as:

$$\begin{array}{ll} \min & \bar{c}_N^T x_N \\ \text{s.t.} & \bar{A} x_N \le \bar{b} \\ & x_N \ge 0, \end{array}$$

where

$$\bar{A} = A_B^{-1} A_N, \quad \text{and} \quad \bar{b} = A_B^{-1} b \ge 0.$$

Note that this program is equivalent to the original primal. Set $x_B = \bar{b} - \bar{A} x_N$.

We saw last time that $c_B = 0$. We considered different cases. If $\bar{c} \ge 0$, then $x_N = 0$ is an optimal solution to this new LP and therefore x is an optimal solution to the original primal (see also Observation 1).

In the case that $\bar{c} \not\ge 0$, there exists a $j \in N$, such that $c_j < 0$. This means that if we increase x_j and keep all other variables in x_N set to zero, we can decrease the value of the solution. How much can we increase x_j? We need to keep the solution feasible, that means that we need to maintain the constraints $\bar{A}_{ij} x_j \le \bar{b}_i$ for all i.

If $\bar{A}_{ij} \leq 0$ for all i, then as we increase x_j, x remains feasible, so as $x_j \to \infty$, the value of the solution goes to minus infinity. Therefore the LP is unbounded.

Now suppose there exists some i such that $\bar{A}_{ij} > 0$. Then x_j can be no larger than $\frac{\bar{b}_i}{\bar{A}_{ij}}$ for any i where $\bar{A}_{ij} > 0$. Thus we increase x_j to the maximum feasible value:

$$\epsilon = \min_{i:\bar{A}_{ij}>0} \frac{\bar{b}_i}{\bar{A}_{ij}}.$$

This is called the *ratio test*. Let i^* be the index that achieves the minimum ratio. Recall $x_B = \bar{b} - \bar{A}x_N$. Setting $x_j = \epsilon$ implies that some variable in x_B will be driven down to 0. In other words, after increasing x_j as much as possible, we will have $x_{i^*} = 0$ for some $i^* \in B$.

Since we now have the same number of variables set to 0 as when we began, this suggests that we have moved to a new "basis"

$$\hat{B} = B - \{i^*\} \cup \{j\}.$$

We will show in the next section that $\hat{B}$ is indeed a basis. From this new basis, we get a new associated solution $\hat{x}$. We will also show that this $\hat{x}$ is exactly the solution we constructed by increasing x_j as much as possible. The process of switching bases is called "making a pivot". Repeatedly doing this gives us an algorithm for solving LPs, called the simplex method.

2 Proving the Simplex Method

Let $\hat{x}$ be the new solution found by the method described above, i.e. $\hat{x}_j = \epsilon$, $\hat{x}_k = 0$ for all $k \in N$, $k \neq j$, and $\hat{x}_B = \bar{b} - \bar{A}\hat{x}_N$. Now we want to prove that the simplex method, under some mild conditions, leads to the optimal solution. In order to do this we are going to show 5 claims:

(i) $c^T\hat{x} \leq c^T x$, i.e. the new solution is not worse than the old solution;

(ii) If x is nondegenerate, then $\epsilon > 0$, i.e. we make progress in our algorithm;

(iii) If all basic feasible solutions are nondegenerate then the simplex method is finite, i.e. we make enough progress and the algorithm terminates in finite time;

(iv) Updated basis $\hat{B}$ after a pivot is indeed a basis;

(v) $\hat{x}$ is the unique solution corresponding to $\hat{B}$.

Claim 2

$$c^T\hat{x} \leq c^T x$$

Proof: By Lemma 1 the LP with objective function $c^T x$ and the LP with objective function $\bar{c}^T x$ and the same constraints are equivalent LPs. Moreover we know that $\bar{c}^T\tilde{x} = c^T\tilde{x} + C$, where C is some constant and $\tilde{x}$ is a feasible solution for both LPs. Therefore it is enough to show $\bar{c}^T\hat{x} \leq \bar{c}^T x$; i.e. we have

$$\bar{c}^T\hat{x} \leq \bar{c}^T x \Rightarrow c^T\hat{x} \leq c^T x.$$

We have

$$\begin{aligned}\bar{c}^T\hat{x} = \bar{c}_B^T\hat{x}_B + \bar{c}_N^T\hat{x}_N &= 0 + \bar{c}_j\epsilon \\ &\leq \bar{c}^T x = \bar{c}_B^T x_B + \bar{c}_N^T x_N = 0,\end{aligned}$$

where the inequality holds because by assumption $\bar{c}_j < 0$ and $\epsilon \geq 0$.

In fact, if $\epsilon > 0$, then the inequality holds strict: $\bar{c}^T \hat{x} < \bar{c}^T x$. $\square$

This means that at least our solution becomes not worse by applying the simplex method, but do we in fact make progress? For this we need to recall the definition of degeneracy: A basic solution x is *degenerate* if there are more than n constraints met with equality.

Claim 3 *If x is nondegenerate, then $\epsilon > 0$.*

Proof: Since x is a nondegenerate basic solution we know that *exactly* n constraints are met with equality. This means $x_j > 0$ for all $j \in B$, i.e. $x_B > 0$. Recall

$$x_B = A_B^{-1} b = \bar{b} > 0,$$

then

$$\frac{\bar{b}_i}{\bar{A}_{ij}} > 0$$

for all i with $\bar{A}_{ij} > 0$, therefore $\epsilon > 0$. $\square$

Let us for now assume that all basic feasible solutions are nondegenerate. We will treat the case of degenerate solutions in a later lecture. So far, we know that we make progress with the simplex method, but do we make enough progress?

Claim 4 *If all basic feasible solutions are nondegenerate then the simplex method is finite.*

Proof: We know from the previous claim that the objective function improves at every pivot and that each basis implies a unique solution, so we will see each basis at most once:

$$\#\text{pivots} \ \leq \ \#\text{bases} \ \leq \binom{n}{m}.$$

We still need to show that $\hat{B}$ after performing a pivot is still a basis. We are going to show this in the next claim. $\square$

Claim 5 *New $\hat{B}$ after a pivot is a basis.*

Proof: By definition of a basis, $\hat{B}$ is a basis if and only if $A_{\hat{B}}$ has full rank. To get $A_{\hat{B}}$ we substituted the j-th column of A for the i^*-th column into A_B.

$$\begin{aligned}
A_{\hat{B}} &= \left[\ \text{old columns} \ \middle|\ A_j \ \middle|\ \text{old columns}\ \right] \\
&= A_B \left[\begin{array}{ccc|c|ccc}
1 & 0 & \cdots & & \cdots & 0 & 0 \\
0 & 1 & & & & 0 & 0 \\
\vdots & \vdots & \ddots & A_B^{-1} A_j & \ddots & \vdots & \vdots \\
0 & 0 & & & & 1 & 0 \\
0 & 0 & \cdots & & \cdots & 0 & 1
\end{array}\right] \\
&\quad \text{(recall that } \bar{A} = A_B^{-1} A_N\text{)}, \\
&= A_B \left[\begin{array}{ccc|c|ccc}
1 & 0 & \cdots & \bar{A}_{1j} & \cdots & 0 & 0 \\
0 & 1 & & \bar{A}_{2j} & & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & & \bar{A}_{(n-1)j} & & 1 & 0 \\
0 & 0 & \cdots & \bar{A}_{nj} & \cdots & 0 & 1
\end{array}\right].
\end{aligned}$$

B is a basis so A_B is non-singular. In order to show that $A_{\hat{B}}$ is non-singular we need to show that the big matrix on the right-hand-side is non-singular. But we chose i^* in the ϵ ratio such that $\bar{A}_{i^*j} > 0$ and therefore the matrix is non-singular and $\hat{B}$ is a basis. $\square$

Claim 6 *$\hat{x}$ is the unique solution corresponding to $\hat{B}$.*

Proof: Note $\hat{x}_k = 0$ for all $k \notin \hat{B}$ (i.e. for all $k \in N - \{j\} \cup \{i^*\}$). We have

$$\bar{A}\hat{x}_N + \hat{x}_B = \bar{A}(\epsilon e_j) + (\bar{b} - \bar{A}(\epsilon e_j)) = \bar{b}.$$

Recall that $\bar{A} = A_B^{-1}A_N$, $\bar{b} = A_B^{-1}b$, and therefore

$$A_N\hat{x}_N + A_B\hat{x}_B = b$$

and we get $A\hat{x} = b$, $\hat{x} \geq 0$, so $\hat{x}$ is a feasible solution with corresponding basis $\hat{B}$. $\square$

3 Some Issues

There are some issues we have to address when using the simplex method. We will go over these issues in the coming lectures.

(i) What happens in the case of degenerate solutions?

   (a) During the simplex method the case of *cycling* may encounter; i.e. the case where we keep revisiting the same basis and the objective function does not decrease.

(ii) Starting point: We need a basic feasible solution to begin our algorithm.

(iii) Assume we are in the case where $\bar{c} \ngeq 0$, i.e. there exists j such that $\bar{c}_j < 0$. Which one of these c_j's do we choose?

   (a) First j such that $\bar{c}_j < 0$?

   (b) j that gives the most improvement?

   (c) j such that $\bar{c}_j$ most negative?

(iv) How much work is involved in every pivot step?