ORIE 6300 Mathematical Programming I	September 23, 2008
Lecture 7	
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We are now finally almost able to prove strong duality. We will first need to show two lemmas before we are able to do this.

Theorem 1 (Farkas' Lemma) *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times 1}$. Then exactly one of the following two condition holds:*

(1) $\exists\, x \in \mathbb{R}^{n \times 1}$ *such that* $Ax = b$, $x \geq 0$;

(2) $\exists\, y \in \mathbb{R}^{1 \times m}$ *such that* $A^T y \geq 0$, $y^T b < 0$.

Proof: First we show that we can't have both (1) and (2). Note that $y^T Ax = y^T(Ax) = y^T b < 0$ since by (1), $Ax = b$ and by (2) $y^T b < 0$. But also $y^T Ax = (y^T A)x = (A^T y)^T x \geq 0$ since by (2) $A^T y \geq 0$ and by (1) $x \geq 0$.

Now we must show that if (1) doesn't hold, then (2) does. To do this, let $v_1, v_2, \ldots, v_n$ be the columns of A. Define

$$Q = cone(v_1, \ldots, v_n) \equiv \{s \in \Re^m : s = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \forall i\}.$$

This is a conic combination of the columns of A, which differs from a convex combination since we don't require that $\sum_{i=1}^{n} \lambda_i = 1$. Then $Ax = \sum_{i=1}^{n} x_i v_i$, there exists an x such that $Ax = b$ and $x \geq 0$ if and only if $b \in Q$.

So if (1) does not hold then $b \notin Q$. We show that condition (2) must hold. We know that Q is nonempty (since $0 \in Q$), closed, and convex, so we can apply the separating hyperplane theorem. The theorem implies that there exists $\alpha \in \Re^m$, $\alpha \neq 0$, and β such that $\alpha^T b > \beta$ and $\alpha^T s < \beta$ for all $s \in Q$. Since $0 \in Q$, we know that $\beta > 0$. Note also that $\lambda v_i \in Q$ for all $\lambda > 0$. Then since $\alpha^T s < \beta$ for all $s \in Q$, we have $\alpha^T(\lambda v_i) \in Q$ for all $\lambda > 0$, so that $\alpha^T v_i < \beta/\lambda$ for all $\lambda > 0$. Since $\beta > 0$, as $\lambda \to \infty$, we have that $\alpha^T v_i \leq 0$. Thus by setting $y = -\alpha$, we obtain $y^T b < 0$ and $y^T v_i \geq 0$ for all i. Since the v_i are the columns of A, we get that $A^T y \geq 0$. Thus condition (2) holds. □

We will also need a similar result, which follows from Farkas' Lemma.

Theorem 2 (Farkas' Lemma′) *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times 1}$. Then exactly one of the following two condition holds:*

(1′) $\exists\, x \in \mathbb{R}^{n \times 1}$ *such that* $Ax \leq b$;

(2′) $\exists\, y \in \mathbb{R}^{1 \times m}$ *such that* $A^T y = 0$, $y^T b < 0$, $y \geq 0$.

The following condition is equivalent to (2′):

(2′′) $\exists\, y \in \mathbb{R}^{1 \times m}$ *such that* $yA = 0$, $y^T b = -1$, $y \geq 0$.

Proof: First we prove that (2′) if and only if (2″). Clearly if (2″) is true, then (2′) is true. If (2′) is true, let $\hat{y} = -\frac{1}{y^T b} y$. Then $\hat{y} \geq 0$ since $y \geq 0$ and $y^T b < 0$. Also

$$\hat{y}^T b = -\frac{y^T b}{y^T b} = -1,$$

and

$$A^T \hat{y} = \frac{-1}{y^T b}(A^T y) = 0.$$

As before, we cannot have both (1′) and (2′). Suppose otherwise. Then

$$y^T Ax = y^T(Ax) \leq y^T b < 0,$$

since $Ax = b$ and $y^T b < 0$, and also

$$y^T Ax = (y^T A)x = (A^T y)^T x = 0,$$

since $A^T y = 0$.

Now suppose (2′) does not hold, so (2″) does not hold either. Define Rewrite the system $A^T y = 0$, $y^T b = -1$ as:

$$\bar{A} = \begin{bmatrix} A^T \\ b^T \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}.$$

Then since (2″) holds, there does not exist $z \in \Re^m$ such that $z \geq 0$ and $\bar{A}z = \bar{b}$. This is just a rewriting of condition (1) of the original Farkas' Lemma such that (1) does not hold. Therefore condition (2) must hold, which implies that there exists s such that $\bar{A}^T s \geq 0$ and $\bar{b}^T s < 0$. Set

$$s = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

for $x \in \Re^n$ and $\lambda \in \Re$. Then $\bar{b}^T s < 0$ implies that

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} < 0,$$

which implies that $\lambda > 0$. Also, $\bar{A}^T s \geq 0$ implies that

$$\begin{bmatrix} A^T \\ b^T \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0,$$

which implies that

$$\begin{bmatrix} A & b \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0,$$

or that $Ax + \lambda b \geq 0$, or that $Ax \geq -\lambda b$, or that $A(\frac{-x}{\lambda}) \leq b$. Therefore $-x/\lambda$ satisfies (1′), so that (1′) holds. □

We are finally, finally ready to prove strong duality. Consider these LPs:

Primal		Dual	
max	$c^T x$	min	$y^T b$
s.t.	$Ax \leq b$	s.t.	$A^T y = c$
			$y \geq 0$

Theorem 3 (Strong Duality) *There are four possibilities:*

1. *Both primal and dual have no feasible solutions (are* infeasible*).*
2. *The primal is infeasible and the dual unbounded.*
3. *The dual is infeasible and the primal unbounded.*
4. *Both primal and dual have feasible solutions and their values are equal.*

Proof: We will show on a problem set that (1) is possible. So let's assume that (1) is not true. There are three remaining cases:

Case 1 Let $\bar{y}$ be a feasible solution for the dual and assume the primal is infeasible. Using Farkas' Lemma$'$, ($1'$) does not hold, so that ($2'$) must hold. Then there exists $\hat{y}$ such that $A^T\hat{y} = 0$, $\hat{y}^T b < 0$, and $\hat{y} \geq 0$. Consider the ray defined by $\bar{y} + \lambda\hat{y}$, $\lambda \geq 0$. Then

$$(\bar{y} + \lambda\hat{y})A = c + \lambda \cdot 0 = c,$$

so that $\bar{y} + \lambda\hat{y}$ is dual feasible. Also,

$$(\bar{y} + \lambda\hat{y})^T b = \bar{y}^T b + \lambda\hat{y}^T b.$$

Since $\hat{y}^T b < 0$, as $\lambda \to \infty$, the value of $\bar{y} + \lambda\hat{y} \to -\infty$. Thus the dual is unbounded.

Case 2 Let $\bar{x}$ be a feasible solution for the primal and assume the dual is infeasible, so that there does not exist y such that $A^T y = c$, $y \geq 0$. Using the original Farkas' Lemma, (1) does not hold (rewriting things a bit), so (2) must hold, which implies there exists an $\hat{x}$ such that $A\hat{x} \geq 0$, $c\hat{x} < 0$. Consider $\bar{x} - \lambda\hat{x}$ for $\lambda \geq 0$. Then

$$A(\bar{x} - \lambda\hat{x}) \leq b - \lambda A\hat{x} \leq b,$$

so $\bar{x} - \lambda\hat{x}$ is primal feasible for $\lambda \geq 0$. Also

$$c^T(\bar{x} - \lambda\hat{x}) = c^T\bar{x} - \lambda c^T\hat{x}.$$

Since $c^T\hat{x} < 0$, as $\lambda \to \infty$, the value of $\bar{x} - \lambda\hat{x}$ goes off to ∞. Thus the primal LP is unbounded.

Case 3 Let $\bar{x}$ and $\bar{y}$ be feasible solutions to the primal and dual, respectively. By weak duality, $c^T\bar{x} \leq \bar{y}^T b$, so the dual is bounded. Let γ be the optimal value of the dual. Suppose that the optimal value of the primal were less than γ:

$$\Rightarrow \nexists x \text{ s.t. } Ax \leq b, \quad cx \geq \gamma$$

$$\Leftrightarrow \nexists x \text{ s.t. } \begin{bmatrix} A \\ -c^T \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \leq \begin{bmatrix} b \\ \gamma \end{bmatrix}$$

Then using Farkas' Lemma$'$, $(1')$ does not hold, so that $(2')$ must hold. Thus there exists a row vector $y \geq 0$ and a scalar $\lambda \geq 0$ such that:

$$\begin{bmatrix} A \\ -c \end{bmatrix}^T \begin{bmatrix} y \\ \lambda \end{bmatrix} = 0, \quad \begin{bmatrix} b \\ \gamma \end{bmatrix}^T \begin{bmatrix} y \\ \lambda \end{bmatrix} < 0$$

Suppose $\lambda = 0$. Then $yA = 0$, $yb < 0$, and $y \geq 0$. Use $(2') \Rightarrow \neg(1')$ on the vector y, which implies that there does not exist x such that $Ax \leq b$. The primal is feasible, so this is a contradiction and in fact $\lambda > 0$. Expanding out the above matrix equation:

$$yA - \lambda c = 0 \quad \Rightarrow \quad \left(\frac{y}{\lambda}\right) A = c$$

Also $\frac{y}{\lambda} \geq 0$, so $\frac{y}{\lambda}$ is a feasible solution. However, $yb - \lambda\gamma < 0$, so $\left(\frac{y}{\lambda}\right) b < \gamma$, which contradicts the optimality of γ.

$\square$