

Recitation 3

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Topic: Feasibility and Unboundedness of LP

Feasibility and Unboundedness ¹

Consider a linear program in arbitrary form. We know that it can potentially be infeasible or have unbounded optimal objective. Additionally, if it's feasible and not unbounded, we can show the existence of an optimal solution by applying the Weierstrass Theorem. Hence, this gives three options for the types of solutions a linear program can have.

Additionally, the dual of a linear program is itself a linear program, so the same three options apply. Hence, the first natural question is what combinations of these can appear for a primal-dual pair of linear programs?

$P \setminus D$	I	O	U
I	?	?	?
O	?	?	?
U	?	?	?

Let's try to fill in some of these boxes.

First, recall the weak duality theorem: If x is a feasible solution to a minimization linear program and y is a feasible solution to its dual, then $b^T y \leq cx$.

Suppose the primal minimization program is unbounded. This immediately implies that the dual must be infeasible. Similarly, if the dual is unbounded, this immediately implies that the primal must be infeasible. To see this in the first case, let y be any feasible solution to the dual. Since the primal is unbounded, there exists an \hat{x} such that $c\hat{x} < b^T y$, contradicting the Weak Duality Theorem. Hence, no such y exists. The other argument can be proved identically.

Hence, our table now looks like:

$P \setminus D$	I	O	U
I	?	?	✓
O	?	?	X
U	✓	X	X

¹Based on previous notes of Maurice Cheung

Given the above theorem, it seems natural to ask whether the reverse implication holds. Does primal infeasibility imply dual unboundedness? Consider the following LP:

$$\begin{aligned} \max \quad & [2, -1] x \\ \text{s.t.} \quad & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ & x \geq 0 \end{aligned}$$

Its corresponding dual is:

$$\begin{aligned} \max \quad & [-1, 2] y \\ \text{s.t.} \quad & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} y \leq \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ & y \geq 0 \end{aligned}$$

Note that the primal is infeasible and that the dual feasible region is exactly the primal feasible region, hence, both are infeasible. This adds another option to our table, giving:

Finally, using Strong Duality Theorem we know when one of primal or the dual has an optimal solution, they both must have an optimal solution. Hence our table looks like:

$P \setminus D$	I	O	U
I	✓	X	✓
O	X	✓	X
U	✓	X	X

Farkas Lemma and its Application

First recall the Farkas' Lemma :

Theorem 1 (Farkas' Lemma) *If $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then exactly one of the following holds:*

1. $\exists x \geq 0$ such that $Ax = b$
2. $\exists y$ such that $A^T y \geq 0, b^T y < 0$

Here is another form of the Farkas Lemma:

Theorem 2 (Alternative Farkas' Lemma) *Exactly one of the following holds:*

1. $\exists x \geq 0$ such that $Ax \leq b$
2. $\exists y$ such that $A^T y \geq 0, b^T y < 0, y \geq 0$

Proof: Note $Ax \leq b, x \geq 0$ is feasible if and only if $Ax + s = b, x, s \geq 0$ is feasible. Apply the original Farkas Lemma to this new system. \square

Let's see an application of the Farkas Lemma. Note that we can only prove that unboundedness implies infeasibility for linear programs and not the converse in the previous section. We now prove a related implication for the unboundedness of feasible **regions** of linear programs.

Theorem 3 (Clark's Theorem) *Given the following primal and dual LPs, if one of them is feasible, then the feasible region for one of them is non-empty and unbounded.*

<i>Primal LP:</i> $\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$	<i>Dual LP:</i> $\begin{aligned} \max \quad & y^T b \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$
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It's important to note that the result of the theorem is that the feasible region of one of the LPs is unbounded, but it may not be the case that the LP has unbounded objective function value with the given objective function.

Proof: There are three possibilities to consider.

1. The primal is infeasible and the dual is unbounded.
2. The dual is infeasible and the primal is unbounded.
3. Both the primal and dual are feasible and not unbounded (hence have optimal solution).

In the first two cases, we immediately have the result we want. Hence, suppose we're in the last case. Now, let $\hat{c} = [-1, -1, \dots, -1]$ and consider the following systems:

1. $\exists \hat{y}$ such that $A^T \hat{y} \leq \hat{c}, \hat{y} \geq 0$
2. $\exists \hat{x}$ such that $A\hat{x} \geq 0, \hat{c}^T \hat{x} < 0, \hat{x} \geq 0$

Alternative Farkas' lemma tells us exact one of them holds.

If (2) holds, let \hat{x} be a feasible solution to (2) and x be a feasible solution to primal LP and $\lambda > 0$. Note \hat{x} is not 0 because $\hat{c}^T \hat{x} < 0$. Then

$$A(x + \lambda \hat{x}) = Ax + \lambda A\hat{x} \geq b + \lambda * 0 = b$$

So $x + \lambda \hat{x}$ is feasible for all $\lambda \geq 0$. Feasible region for primal LP is unbounded and we're done.

Otherwise if (1) holds, let \hat{y} be a feasible solution to (1) (note \hat{y} is not 0 because $A^T \hat{y} \leq \hat{c}$) and let y be a feasible solution to dual LP. Then using a similar argument as above, we can show that for any $\lambda \geq 0$, $y + \lambda \hat{y}$ is feasible for the dual, which shows the feasible region for dual LP is unbounded and we're done. \square