We have seen a constraint-generation procedure to aid in solving the LP relaxation of the traveling salesman problem. This week we will consider another method for producing a bound on the optimal IP value.\footnote{Based on previous notes of Maurice Cheung}

In general, consider an integer program of the form:
\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad Dx \geq d \\
& \quad x \quad \text{integer}
\end{align*}
\]
Assume all of the input data is integer-valued and suppose we can “quickly” optimize over the set
\[X = \{ x : Dx \geq d, \ x \text{ integer} \} .\]

Now, we will relax the “hard” constraints \( Ax \geq b \) by removing them and inserting a penalty for violations. Let \( p \) be a non-negative vector in \( \mathbb{R}^n \), and consider the new problem:
\[
\begin{align*}
\min & \quad cx + p(b - Ax) \\
\text{s.t.} & \quad x \in X.
\end{align*}
\]
Let \( Z(p) \) be the optimal objective value of this LP. Clearly, \( Z(p) \) is no larger than the optimal IP value since the optimal IP solution is feasible for this problem and it satisfies \( Ax^*_IP \geq b \), so \( p(b - Ax^*_IP) \leq 0 \).

**The Lagrangean Dual**

For any \( p \geq 0 \), we have \( Z(p) \leq Z_{IP} \), giving us a bound on the optimal IP value. To get the best possible bound, consider the problem:
\[
Z_D = \max_{p \geq 0} Z(p) = \max_{p \geq 0} \min_{x \in X} cx + p(b - Ax)
\]
If \( X = \{ x^1, \ldots, x^k \} \) is a finite set, then we can compute the value \( Z_D \) by the following LP:
\[
\begin{align*}
\max & \quad q \\
\text{s.t.} & \quad q \leq cx^i + p(b - Ax^i) & i = 1, \ldots, k \\
& \quad p \geq 0
\end{align*}
\]
Note that when \( X \) is large, this is inefficient. However taking the dual of this LP, we get:
\[
\begin{align*}
\min & \quad \sum_j y_j(cx^j) \\
\text{s.t. :} & \quad \sum_j y_j(Aix^j - b_i) \geq 0 \quad \forall i = 1, \ldots, m \\
& \quad \sum_j y_j = 1 \\
& \quad y \geq 0
\end{align*}
\]
If we rearrange the equations, and use the fact that $\sum_j y_j = 1$, we get an equivalent representation:

$$\begin{align*}
\min_{c} & \quad c \left( \sum_j y_j x^j \right) \\
\text{s.t.} & \quad A \left( \sum_j y_j x^j \right) \geq b \\
& \quad \sum_j y_j = 1 \\
& \quad y \geq 0
\end{align*}$$

Letting $\text{conv}(X)$ be the convex hull of $X$, note that $x \in \text{conv}(X)$ iff $x = \sum_j \alpha_j x^j$, $\sum_j \alpha_j = 1$, $\alpha \geq 0$, $x^j \in X$. Hence, this LP is exactly the same as optimizing over the convex hull of $X$. Hence, this can be written as:

$$\begin{align*}
\min & \quad cx \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x \in \text{conv}(X)
\end{align*}$$

Hence, $Z_D$ can be computed by solving this LP. Note that an immediate corollary of this is that $Z_{LP} \leq Z_D$ since $\text{conv}(X) \subseteq \{ x : Dx \geq d \}$. That means the bound provided by the Lagrangean dual is as least as strong as the LP-bound. Additionally, note that the analysis here can be extended to the case where $X$ is not finite.

**The Held-Karp Bound for Traveling Salesman Problem**

Consider the TSP, that is, the problem of finding a minimum cost tour in a graph. In a previous recitation, we used the following characterization of tours: A subgraph forms a tour iff each vertex has degree 2 and each cut has at least 2 edges crossing it. This led to an LP relaxation that could be used to give a bound on the value of the optimal tour. Here, we will use a slightly different characterization. Number the nodes in the graph 1 through $n$ for some arbitrary numbering.

**Definition 1** A subgraph is a 1-tree if it is a spanning tree on nodes $\{2, \ldots, n\}$ along with two edges incident to node 1.

**Claim 1** A subgraph is a tour iff it is a 1-tree where each vertex $2, \ldots, n$ has degree 2.

Thus, the following IP solves for the minimum cost tour, where $V_{-1}$ is the set of vertices $\{2, \ldots, n\}$, and $E(S)$ is the set of edges with both endpoints in $S$:

$$\begin{align*}
\min & \quad \sum_{e} c_e x_e \\
\text{s.t.:} & \quad \sum_{e \in E(V_{-1})} x_e = n - 2 \\
& \quad \sum_{e \in E(S)} x_e \leq |S| - 1 \quad \forall S \subset V_{-1} \\
& \quad \sum_{e \in \delta(\{1\})} x_e = 2 \\
& \quad \sum_{e \in \delta(\{i\})} x_e = 2 \quad \forall i \in V_{-1} \\
& \quad x \text{ integer}
\end{align*}$$

Let $X$ be the set of vectors that satisfy all of these constraints except for the vertex constraints for $i \in V_{-1}$. These constraints say that there must be $n - 2$ edges in $E(V_{-1})$ and any subgraph on $k$ nodes in $V_{-1}$ can have at most $k - 1$ edges. Note that this implies that there are no cycles on that subgraph, therefore it must be a tree. Since we also require that there are two edges incident to node 1, the set $X$ is exactly the set of 1-trees.
Thus, we can consider dualizing the vertex constraints for vertices in $V_{-1}$. The bound $Z_D$ obtained this way is known as the Held-Karp lower bound. This bound is actually fairly easy to calculate. Since the set $X$ is the set of 1-trees, we can calculate the min cost 1-tree for a particular vector $p$ without solving an LP at all.

In this case, it turns out that the polytope defined by the non-dualized constraints has integer extreme points. Using the notation from the previous section, this says that $\{x : Dx \geq d\} = conv(X)$. Thus, for this formulation, $Z_{LP} = Z_D$. So the value of the Held-Karp lower bound is not in the strength of the bound, but in the efficient computation.