1 Conic Programming and Strong Duality

Recall from last time on primal/dual pair of conic programming

\[
\begin{align*}
\inf & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
\sup & \quad b^T y \\
\text{s.t.} & \quad A^T y + s = c
\end{align*}
\]

\[x \in K \quad s \in K^* \equiv \{s \in \mathbb{R}^n, \quad s^T x \leq 0, \quad \forall x \in K\}\]

Various weirdness emerge even from “nice” cones (like SOC):

- Weak duality holds;
- Strong duality may not;
- Maybe no optimal solution (hence inf/sup)

Today we work out some conditions under which strong duality holds. Recall we showed the analog of Farkas’ Lemma doesn’t hold. Both \(Ax = b, \ x \in K\) and \(-A^T y \in K^*, b^T y > 0\) can be infeasible. So let’s see how we can modify the condition so we can get something.

**Definition 1** \(Ax=b, \ x \in K\) is asymptotically feasible if \(\forall \epsilon > 0. \ \exists \Delta b, \|\Delta b\| < \epsilon\) such that \(Ax = b + \Delta b, x \in K\) is feasible.

**Theorem 1** (Asymptotic Farkas' Lemma) Either \(Ax = b, x \in K\) is asymptotically feasible, or \(-A^T y \in K^*, b^T y > 0\) is feasible, but not both.

**Proof:** Let \(Q = \{\tilde{b} \in \mathbb{R}^m : \exists x \in K \text{ s.t. } Ax = \tilde{b}\}\), Then \(b \in cl(Q)\) iff \(Ax = b, x \in K\) asymptotically feasible (where \(cl(Q)\) is the closure of \(Q\)). If \(Ax = b, x \in K\) not asymptotically feasible, then \(b \not\in cl(Q)\). Since \(cl(Q)\) is closed, nonempty \((0 \in Q)\), and convex, we can apply the separating hyperplane theorem. So \(\exists y \in \mathbb{R}^n, \beta\) such that \(y^T b > \beta\) and \(y^T \tilde{b} < \beta\) for all \(\tilde{b} \in cl(Q)\). Since \(0 \in Q, \beta > 0\). Thus

\[
\begin{align*}
y^T b > 0 & \quad \text{and} \quad y^T (Ax) < \beta, \quad \forall x \in K \\
\iff y^T (Ax) < \beta, \quad \forall x \in K, \quad \lambda > 0 \\
\iff y^T (Ax) < \beta/\lambda, \quad \forall x \in K, \quad \lambda > 0 \\
\iff y^T (Ax) \leq 0, \quad \forall x \in K \\
\iff x^T (A^T y) \leq 0, \quad \forall x \in K \\
\iff -A^T y \in K^*, \quad \forall x \in K
\end{align*}
\]
by the definition of $K^*$.

Now we need to define the primal value under asymptotic feasibility.

**Definition 2** \(a\text{-opt} = \lim_{\varepsilon \to 0} \inf_{\Delta b} \{c^T x : x \in K, Ax = b + \Delta b\}\) (i.e. limiting value of asymptotically feasible solution)

**Theorem 2** If primal is asymptotically feasible, then a-opt equals dual optimal.

**Proof:** Consider the following system:

\[
\begin{bmatrix}
A & 0 & 1 \\
c^T & 1
\end{bmatrix}
\begin{bmatrix} x \\ z \end{bmatrix} =
\begin{bmatrix} b \\ \lambda \end{bmatrix}, \quad \begin{bmatrix} x \\ z \end{bmatrix} \in K \times \mathbb{R}^2
\]  

\[\iff Ax = b, \; c^T x + z = \lambda; \; z \geq 0 \text{ (i.e. } c^T x \leq \lambda)\]

By asymptotic Farkas Lemma, either the system above is asymptotically feasible, and thus a-opt \(\leq \lambda\) or the system

\[-\begin{bmatrix}
A & 0 & 1 \\
c^T & 1
\end{bmatrix}^T \begin{bmatrix} y \\ \gamma \end{bmatrix} \in (K \times \mathbb{R}^2)^* = K^* \times \mathbb{R}^2
\]

\[\begin{bmatrix} b \\ \lambda \end{bmatrix}^T \begin{bmatrix} y \\ \gamma \end{bmatrix} > 0\]

is feasible which means that there exists \(y, \gamma\) such that

\[A^T y - \gamma c \in K^*\]

\[-\gamma \geq 0\]

\[b^T y > -\lambda \gamma\]  

(\text{**})

First, suppose that there is \(\lambda\) such that a-opt \(\leq \lambda < \text{dual-optimal}\). Then there exists a dual feasible \(y\) such that \(b^T y > \lambda, \; -A^T y + c \in K^*\). But then \(\begin{bmatrix} y \\ -1 \end{bmatrix}\) is feasible for (\text{**}), a contradiction to a-opt \(\leq \lambda\). Thus a-opt \(\geq \text{dual-optimal}\).

Second, suppose that there is a \(\lambda\) such that dual-opt \(< \lambda <\text{a-opt}\). Then (\text{**}) is feasible for same \(\begin{bmatrix} y \\ \gamma \end{bmatrix}\). If \(\gamma = 0\) then \(-A^T y \in K^*, \; b^T y > 0\) is feasible. By the asymptotic Farkas’ Lemma, this implies that \(Ax = b, \; x \in K\) is not asymptotically feasible, which is a contradiction.

Thus we can assume \(\gamma < 0\). Then consider \(\tilde{y} = -\frac{1}{\gamma}y\). By the feasibility of \((y, \gamma)\), we have that

\[-A^T \tilde{y} + c \in K^*\]

\[b^T \tilde{y} > \lambda.\]
This contradicts our hypothesis that dual-optimal $< \lambda$.

We have shown that $a$-opt can be neither less than the dual optimal, nor greater than the dual optimal, and so it must be equal to the dual optimal. □

We can similarly define the asymptotic optimal of dual.

**Definition 3** $a$-dual-opt $= \lim_{\epsilon \to 0} \sup_{\|\Delta c\| < \epsilon} (\sup b^T y : c + \Delta c - A^T y \in K^*)$.

By similar reasoning, we can prove the following theorem.

**Theorem 3** If dual is asymptotic feasible, then $a$-dual-opt = primal opt.

We now want to state conditions under which strong duality holds. We now know that $a$-opt $= \text{dual optimal} \leq a$-dual-opt $= \text{primal optimal}$. When is the inequality an equality? We first need another definition.

**Definition 4** The primal is strongly feasible if $\exists \epsilon > 0$ such that $\forall \Delta b$ with $\|\Delta b\| < \epsilon$, then $Ax = b + \Delta b, x \in K$ is feasible.

The dual is strongly feasible if $\exists \epsilon > 0$ such that $\forall \Delta c$ with $\|\Delta c\| < \epsilon$, then $A^T + S = c + \Delta c$, $s \in K^*$ is feasible.

**Observation 1** If $\exists x$ such that $Ax = b, x \in \text{int } K$, then the primal is strongly feasible.

**Theorem 4** If either primal or dual is strongly feasible, then primal opt = dual opt. (i.e. strong duality holds).

**Corollary 5** Strong duality holds if there exists a feasible primal solution $x \in \text{int } K$, or if there exists a feasible dual solution with $s \in \text{int } K^*$.

**Proof:** Assume primal, dual are both asymptotic feasible, and the dual is strongly feasible. (Note that we are skipping a case in which the primal is infeasible, the dual unbounded). Then, $a$-opt $= \text{dual} \leq a$-dual-opt $= \text{primal}$.

Suppose that $a$-opt $< \text{primal}$. Then, there exists a sequence $\{x_i\} \in K$ and $\{\Delta b_i\}$ such that

$$Ax_i = b + \Delta b_i, \quad \Delta b_i \to 0, \quad c^T x_i \to a\text{-opt}.$$ 

We claim that $\|x_i\| \to \infty$, since otherwise $\{x_i\}$ has convergent subsequence, with limit point $x$ having $x \in K$, $Ax = b$, and $c^T x = a\text{-opt}$. Such an $x$ implies that $a$-opt $= \text{primal}$.

Let $\Delta c$ be a limit point of $\{-\frac{x_i}{\|x_i\|}\}$ so that $\|\Delta c\| = 1$. For given $\epsilon > 0$ consider

$$\min \ (c + \epsilon \Delta c)^T x$$
$$s.t. \ Ax = b$$
$$x \in K.$$
The asymptotic optimal of this instance is at most
\[
\lim_{i} \inf (c + \epsilon \Delta c)^T x_i = \lim_{i} \inf c^T x_i + \epsilon \lim_{i} \inf \Delta c^T x_i
\]
\[
= a_{-\text{opt}} + \epsilon \lim_{i} \inf \Delta c^T x_i
\]
\[
= a_{-\text{opt}} - \epsilon \lim_{i} \|x_i\|
\]
\[
= -\infty.
\]

Since the asymptotic optimal is unbounded, dual must be infeasible; i.e. the following is infeasible:
\[
\sup b^T y
\]
\[
A^T y + s = c + \epsilon \Delta c
\]
\[
s \in K^*
\]
Since \(\epsilon\) can be arbitrarily small, this implies that the original dual is not strongly feasible, a contradiction.

\[\square\]

**Corollary 6** If primal is feasible and dual is strongly feasible, then the primal has an optimal solution.

**Proof:** As above. If the dual is strongly feasible and the primal feasible, then strong duality holds and there exists a feasible sequence \(\{x_i\} \subset K\), \(\Delta b \rightarrow 0\), \(c^T x_i \rightarrow a_{-\text{opt}}\). If \(\{x_i\}\) does not have a convergent subsequence, then \(\|x_i\| \rightarrow \infty\) implies that the dual is not strongly feasible. So there is a convergent subsequence, and the limit point \(x\) is feasible, with \(c^T x = \text{opt}\). \[\square\]