ORIE 6300 Mathematical Programming I November 20, 2014

Lecture 25

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1 NP-Complete Problems and General Strategy

In the last lecture, we defined two classes of problems, **P** and **NP**. While $\mathbf{P} \subseteq \mathbf{NP}$, it is still an open question whether $\mathbf{NP} \subseteq \mathbf{P}$. We recognized a special class of problems inside **NP**, which are called **NP**-complete problems. They are the fundamental problems to tackle in order to solve **P** vs **NP**. We gave three examples of **NP**-complete problems (proof omitted): SAT, Partition, and 3-Partition. Our goal in this lecture is to recognize other **NP**-complete problems based on Partition and SAT problems.

There is a general strategy to show that a problem B is **NP**-complete. The first step is to prove that B is in **NP** (which is usually easy) and the second step is to prove there is an **NP**-complete problem A such that it has a polynomial reduction to problem B. In general, the difficulties lie in the second step.

2 Knapsack

Let us recall the decision version of the Knapsack problem: given n items with size $s_1, s_2, ..., s_n$, value $v_1, v_2, ..., v_n$, capacity B and value V, is there a subset $S \subseteq \{1, 2, .., n\}$ such that $\sum_{i \in S} s_i \leq B$ and $\sum_{i \in S} v_i \geq V$?

Theorem 1 *Knapsack is* ***NP****-complete.*

Proof: First of all, Knapsack is **NP**. The proof is the set S of items that are chosen and the verification process is to compute $\sum_{i \in S} s_i$ and $\sum_{i \in S} v_i$, which takes polynomial time in the size of input.

Second, we will show that there is a polynomial reduction from Partition problem to Knapsack. It suffices to show that there exists a polynomial time reduction $Q(\cdot)$ such that $Q(X)$ is a 'Yes' instance to Knapsack iff X is a 'Yes' instance to Partition. Suppose we are given $a_1, a_2, \ldots, a_n$ for the Partition problem, consider the following Knapsack problem: $s_i = a_i, v_i = a_i$ for $i = 1, \ldots, n$, $B = V = \frac{1}{2} \sum_{i=1}^{n} a_i$. $Q(\cdot)$ here is the process converting the Partition problem to Knapsack problem. It is clear that this process is polynomial in the input size.

If X is a 'Yes' instance for the Partition problem, there exists S and T such that $\sum_{i \in S} a_i = \sum_{i \in T} a_i = \frac{1}{2} \sum_{i=1}^{n} a_i$. Let our Knapsack contain the items in S, and it follows that $\sum_{i \in S} s_i = \sum_{i \in S} a_i = B$ and $\sum_{i \in S} v_i = \sum_{i \in S} a_i = V$. Therefore, $Q(X)$ is a 'Yes' instance for the Knapsack problem.

Conversely, if $Q(X)$ is a 'Yes' instance for the Knapsack problem, with the chosen set S, let $T = \{1, 2, ..n\} - S$. We have $\sum_{i \in S} s_i = \sum_{i \in S} a_i \leq B = \frac{1}{2} \sum_{i=1}^{n} a_i$, and $\sum_{i \in S} v_i = \sum_{i \in S} a_i \geq V = \frac{1}{2} \sum_{i=1}^{n} a_i$. This implies that $\sum_{i \in S} a_i = \frac{1}{2} \sum_{i=1}^{n} a_i$ and $\sum_{i \in T} a_i = \sum_{i=1}^{n} a_i - \frac{1}{2} \sum_{i=1}^{n} a_i = \frac{1}{2} \sum_{i=1}^{n} a_i$.

Therefore, $\{S, T\}$ is the desired partition, and X is a 'Yes' instance for the Partition problem. This establishes the **NP**-completeness of Knapsack problem. □

Remark 1 *In the previous lecture, we showed that Knapsack problem can be solved using dynamic programming with running time $O(n^3B^2)$, where n is the number of items and B is the capacity. Since our input is binary, nB^2 is exponential in the input size ($B = 2^{\log B}$), thus DP does not provide a polynomial running time algorithm.*

On the other hand, if the given input to the Knapsack problem is unary rather than binary (that is, we encode a 5 as 11111), then DP provides a polynomial running time algorithm. We call such algorithms pseudo-polynomial time algorithms.

Hence, we see that Knapsack is not **NP**-complete if the given input is unary (assuming $\mathbf{P} \neq \mathbf{NP}$), but **NP**-complete when the given input is binary. Such problems are called **weakly NP-complete**. However, some problems (like 3-Partition) are **NP**-complete even if the given input is uniary. We call such problems **strongly NP-complete**.

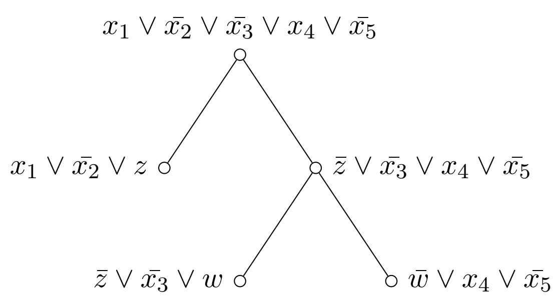
3 3-SAT

The SAT problem is the following: given n boolean variables $x_1, x_2, ..., x_n$, m clauses (e.g. $x_1 \vee \bar{x_3} \vee x_7$), is there an assignment of true/false to the x_i, such that all clauses are satisfied? 3-SAT problem is a special case of SAT problem in the sense that each clause contains at most 3 variables.

Theorem 2 *3-SAT is **NP**-complete.*

Proof: First of all, since 3-SAT problem is also a SAT problem, it is **NP**. We now show that there is a polynomial reduction from SAT to 3-SAT.

Given m clauses in the SAT problem, we will modify each clause in the following recursive way: while there is a clause with more than 3 variables, replace it by two clauses with one new variable. The tree below is an example of this process, and we will use it for the demonstration of proof. The new 3-SAT problem contains all the clauses corresponding to the leaves, they are $x_1 \vee \bar{x_2} \vee z$, $\bar{z} \vee \bar{x_3} \vee w$, and $\bar{w} \vee x_4 \vee \bar{x_5}$.



We first observe that this reduction process is polynomial in the input size. For a clause consisting of k variables, we can build a tree recursively until each leaf is a clause consisting of exactly 3 variables. At i-th level of the tree, the clause corresponds to rightest node at each level contains one less variable than the previous layer. Hence, the tree has $k - 3$ layers in total. This implies that we will construct $k - 2$ new clauses that consists of exactly 3 variables for each clause that consists of k variables in the SAT problem. Suppose the original SAT problem has m clauses, with $k_1, ..., k_m$ variables respectively, we will construct a 3-SAT problem with $\sum_{i=1}^{m}(k_i - 2)$ clauses. And this procedure takes $O(2\sum_{i=1}^{m}(k_i - 2))$ steps, which is polynomial in the size of input.

For the final step, we claim that the original SAT problem is a 'Yes' instance iff the constructed 3-SAT problem is a 'Yes' instance. The key property here is that each step (during the tree construction) maintains satisfiability, i.e, the clauses at level i can be satisfied iff the clauses at level $i+1$ can be satisfied. For a demonstration, we will use the above tree. First suppose that $x_1 \vee \bar{x_2} \vee \bar{x_3} \vee x_4 \vee \bar{x_5}$ is true, then either $x_1 \vee \bar{x_2}$ is true or $\bar{x_3} \vee x_4 \vee \bar{x_5}$ is true. In the previous case, set $z = False$, and in the latter case, set $z = True$. We see that with this assignment, both $x_1 \vee \bar{x_2} \vee z$ and $\bar{z} \vee \bar{x_3} \vee x_4 \vee \bar{x_5}$ are satisfied. Conversely, if both $x_1 \vee \bar{x_2} \vee z$ and $\bar{z} \vee \bar{x_3} \vee x_4 \vee \bar{x_5}$ are satisfied, then if $z = True$, we know that $\bar{x_3} \vee x_4 \vee \bar{x_5}$ is true; if $z = False$, we know that $x_1 \vee \bar{x_2}$ is true. Both imply that the original clause is true.

This property allows us to prove the general claim. For ==> direction, we start from the root of the tree and use the satisfiability property to deduce that all the clauses at the leaves can be satisfied. For <== direction, we start from the leaves and use the satisfiability property to show that the root clause can be satisfied as well. This completes the proof that 3-SAT problem is **NP**-complete. □

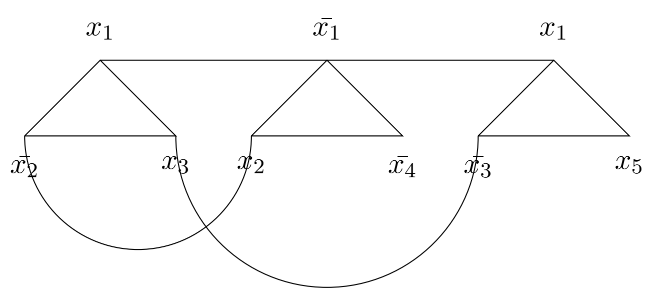
4 Independent Set Problem

Given a graph $G = (V, E)$, an independent set (**IS**) S is a subset of V such that for all $i, j \in S$, $(i, j) \notin E$. The maximum **IS** problem is to find an independent set of maximum size, and the decision version of this problem is the following: give $G = (V, E)$, is there an independent set of size at least B ?

Theorem 3 ***IS** is **NP**-complete.*

Proof: First of all, **IS** is NP with proof S. The verification process consists of checking all possible pairs in S and checking $|S| = B$. It takes $\binom{B}{2} + 1$ steps, which is polynomial in the size of the input.

Secondly, we claim that there is a polynomial-time reduction from 3-SAT problem to **IS** problem. The construction is the following: give a 3-SAT problem with m clauses, we draw m triangles with nodes representing the literals appearing in the clause. Then we connect each node corresponding to a literal x_i with each node corresponding to a literal $\bar{x_i}$, for all i. For example, consider a 3-SAT problem: $x_1 \vee \bar{x_2} \vee x_3$, $\bar{x_1} \vee x_2 \vee \bar{x_4}$, $x_1 \vee \bar{x_3} \vee x_5$, we will convert it to the following graph:



Note that the newly constructed graph G consists of $3m$ nodes, and at most $3m + \binom{3m}{2}$ edges. Hence, the reduction process takes time polynomial in the input size. Moreover, we claim that a

3-SAT problem is a 'Yes' instance iff (G, m) is a 'Yes' instance to the **IS** problem. Suppose that 3-SAT is satisfiable, then for each triangle, we choose one node such that the corresponding literal satisfies the clause (that is, x_i is set true or $\bar{x}_i$ is set false). For any two nodes we choose, they are from two different triangles. If they are connected to each other, they are x_i and $\bar{x}_i$ for some i by our construction. But it is not possible that $x_i = True$ and $\bar{x}_i = True$. Hence, the set of nodes we choose forms an independent set of size m.

Conversely, if we have an independent set of size m for G, then each node must come from different triangles since each triangle is connected. For each node we choose, if it corresponds to variable x_i for some i, we set $x_i = True$; if it corresponds to $\bar{x}_i$ for some i, then we set $x_i = False$; the assignment is consistent because we cannot have both x_i and $\bar{x}_i$ in the independent set because they are joined by an edge. This assignment satisfies the clause corresponding to each triangle. This shows that the 3-SAT instance is satisfiable. $\square$