ORIE 6300 Mathematical Programming I	October 31, 2014
Lecture 19	
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1 The Ellipsoid Method for LP

Recall we discussed the ellipsoid method last time: Given some bounded polyhedron $P = \{x \in \mathbb{R}^n : Cx \leq D\}$, either finds $x \in P$ or states that $P = \emptyset$; that is, P is infeasible. How can we use this feasibility detector to solve an optimization problem such as $\min c^T x : Ax \leq b, x \geq 0$? We claim that we can do this by making three calls to the ellipsoid method.

1.1 Idea of Ellipsoid method

Let's now give the basic idea of how the ellipsoid method will work.

- Start with a sphere large enough to contain all feasible points. Call the sphere E_0, center a_0.
- If $a_k \in P$, done (i.e. obeys constraints). Return a_k.
- If not, $a_k \notin P$, since $C_j a_k > d_j$ for some j.
    - Divide ellipsoid E_k in half through center a_k with a hyperplane parallel to the constraint $C_j x = d_j$.
- Compute new ellipsoid E_{k+1}, center a_{k+1}, containing the "good" half of E_k that contains P. Repeat.

1.2 Showing progress using the Ellipsoid method

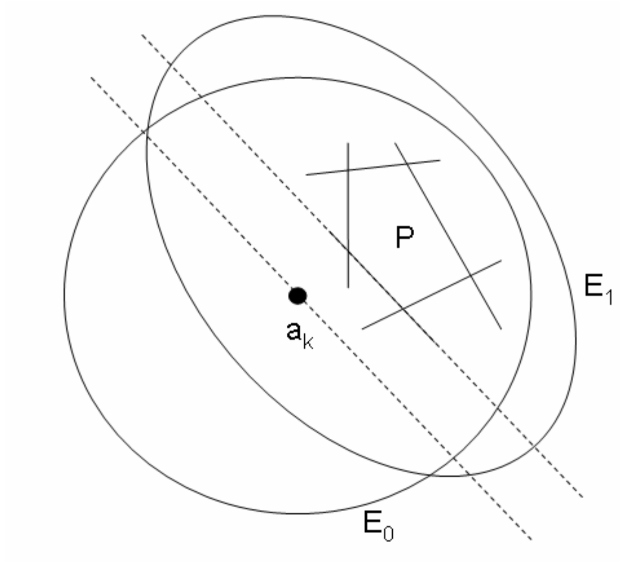
Recall: $L \equiv$ number of bits to represent C, d. For any vertex x, x_j needs at most $nU + n\log(n)$ bits to represent $n \equiv$ numbers of vars, $U \equiv$ largest entry in C, d.

- $\implies |x_j| \leq 2^{nU+n\log(n)}$
- $\implies$ sphere of radius 2^L contains all vertices and has volume $2^{O(nL)}$.

Note, our initial ellipsoid is a sphere centered at origin and we know for any vertex x, $|x_j| \leq 2^{nU+n\log n}$, so sphere of radius 2^L, volume $2^{O(nL)}$, will contain the feasible region.

In order to show progress, we will show:

1. (today) that after any $O(n)$ iterations, the volume of the ellipsoid will be reduced by a factor of ≈ 2
2. (next time) that if P is feasible, then it has a region of volume $2^{\Omega(nL)}$.



Note that these two claims together will imply that the algorithm runs in polynomial time: After $O(n^2L)$ iterations (n per factor of 2, $O(nL)$ factors of 2), either we find a feasible point or the ellipsoid has volume smaller than any feasible region, so P is infeasible.

1.3 Unit sphere split by hyperplane $x_1 > 0$

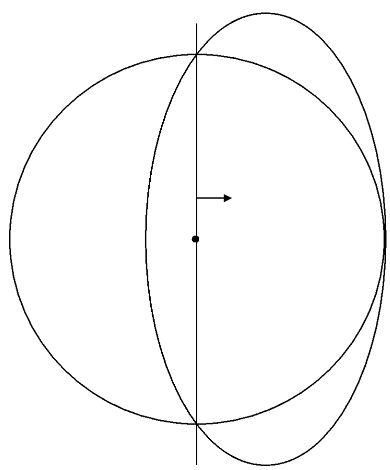


Figure 1: *Unit sphere split by hyperplane* $x_1 > 0$

As a start, consider the unit sphere E_0 centered at the origin with radius 1. Consider dividing E_0 with plane $x_1 \geq 0$. Thus

$$E_0 = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1 \right\}$$

We now consider the ellipsoid

$$E_1 = \left\{ x \in \mathbb{R}^n : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} {x_i}^2 \leq 1 \right\}$$

We need to show that:

1. E_1 contains all points in the intersection of E_0 and $x_1 \geq 0$
2. Volume of E_1 is some factor smaller than E_0.

Lemma 1

$$E_1 \supseteq E_0 \cap \{x : x_1 \geq 0\}$$

Proof: Pick $x \in E_0 \cap \{x : x_1 \geq 0\}$

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} {x_i}^2$$

We know that $\sum_{i=2}^{n} {x_i}^2 \leq 1 - {x_1}^2$ so,

$$\begin{aligned} \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} {x_i}^2 &\leq \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} 1 - {x_1}^2 \\ &= \frac{2n+2}{n^2} {x_1}^2 - \frac{2(n+1)}{n^2} x_1 + 1 \\ &= \frac{2n+2}{n^2} ({x_1}^2 - x_1) + 1 \end{aligned}$$

Since $0 \leq x_1 \leq 1$ because x_1 is inside unit sphere and $x_1 \geq 0 \implies x_1^2 - x_1 \leq 0$

$$\therefore ((2n+2)/n^2)x_1^2 - x_1) + 1 \leq .$$

$\square$

Lemma 2

$$\frac{volume(E_1)}{volume(E_0)} \leq e^{\frac{-1}{2(n+1)}} < 1$$

Proof: First we define the general form of an ellipsoid with center a:

$$E(a, A) = \left\{ x : (x-a)^T A^{-1} (x-a) \leq 1 \right\}$$

where A is a symmetric, positive definite matrix (i.e. $v^T A v > 0 \forall v \in \mathbb{R}^n$). Then for an ellipsoid $E_1 = E(a_1, A)$ with center $a_1 = \frac{1}{n+1} e_1$,

$$A^{-1} = \begin{bmatrix} \left(\frac{n+1}{n}\right)^2 & 0 & \dots & \dots & 0 \\ 0 & \frac{n^2-1}{n^2} & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{n^2-1}{n^2} \end{bmatrix} \text{ and } A = \begin{bmatrix} \left(\frac{n}{n+1}\right)^2 & 0 & \dots & \dots & 0 \\ 0 & \frac{n^2}{n^2-1} & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{n^2}{n^2-1} \end{bmatrix}$$

$$\implies A = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} e_1 e_1^T\right)$$

Fact 1 *Volume of* $E(a, A)$ *is* $\sqrt{det(A)}$ *times the volume of a unit sphere*

$$\begin{aligned} \frac{\text{volume}(E_1)}{\text{volume}(E_0)} &= \sqrt{det(A)} \\ &= \left[\left(\frac{n}{n+1}\right)^2 \left(\frac{n^2}{n^2-1}\right)^{n-1}\right]^{\frac{1}{2}} \end{aligned}$$

We use the fact that $1 + x \leq e^x$ for all x, so that

$$\begin{aligned} \implies \left[\left(\frac{n}{n+1}\right)^2 \left(\frac{n^2}{n^2-1}\right)^{n-1}\right]^{\frac{1}{2}} &\leq \left[e^{\frac{-2}{n+1}} e^{\frac{n-1}{n^2-1}}\right]^{\frac{1}{2}} \\ &= \left[e^{\frac{-2}{n+1}} e^{\frac{1}{n+1}}\right]^{\frac{1}{2}} \\ &= \left[e^{\frac{-1}{n+1}}\right]^{\frac{1}{2}} = e^{\frac{-1}{2(n+1)}} \end{aligned}$$

□

Will show that in general that $\text{volume}(E_{k+1}) \leq e^{\frac{-1}{2(n+1)}} \text{volume}(E_k)$

- $\implies$ after k iterations, volume drops by a factor of at least $e^{\frac{-k}{2(n+1)}}$
- $\implies$ after $2(n + 1)$ iterations, volume drops by a factor of at least $e > 2$.

To deal with the general case, we want to show for any ellipsoid E with center a and constraint $C_j = c$, we can find a new ellipsoid E' with center a' such that $E' \supseteq E \cap x : c^T x \leq c^T a$ and $\text{volume}(E') \leq e^{\frac{-1}{2(n+1)}} \text{volume}(E)$. We now deal with a slightly more complicated case.

1.4 Unit sphere split by arbitrary hyperplane

First, suppose that $E_0 = E(0, I)$, the unit sphere centered at origin, but now we have arbitrary constraint c. Assume $\|c\| = 1$. (*i.e.*, $c^T c = 1$). In order to handle this, the main idea is to reduce to previous case. Consider applying a rotation $y = T(x)$, so that $-e_1 = T(c)$. Then rotate E_1 back using T^{-1}.

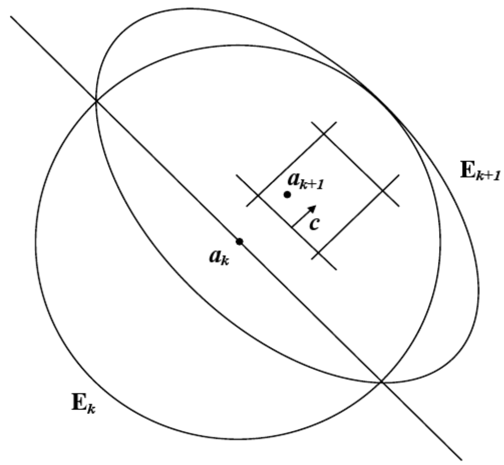


Figure 2: *General Case for Unit Sphere*

Since T is a rotation, $y = T(x) = Ux$ for some orthonormal matrix U (i.e. $U^T = U^{-1}$). We want $Uc = -e_1$, so $c = -U^{-1}e_1 = -U^T e_1$. In the transformed space, the desired ellipsoid is $\{x \in \mathbb{R}^n : (Ux - a)^T A^{-1}(Ux - a) \leq 1\}$. Since $U^T U = I$, this is the same as $\{x : (Ux - a)^T U U^T A^{-1} U U^T (Ux - a) \leq 1\}$.

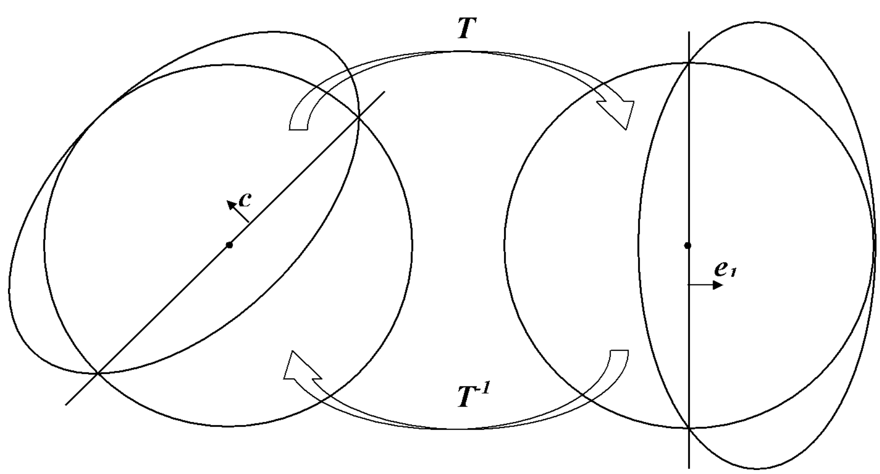


Figure 3: *Rotation*

Now we observe that

$$\begin{aligned}(Ux - a)^T U &= ((Ux)^T - a^T)U \\ &= (x^T U^T - a^T)U \\ &= x^T - a^T U \\ &= (x - U^T a)^T,\end{aligned}$$

and

$$U^T(Ux - a) = x - U^T a,$$

where we define

$$U^T a = U^T \left(\frac{1}{n+1} e_1 \right) = -\frac{1}{n+1} e =: \hat{a}.$$

If we set $\hat{A}^{-1} = U^T A^{-1} U$, then we get

$$\begin{aligned}
\hat{A} &= (U^T A^{-1} U)^{-1} \\
&= U^{-1} A (U^{-1})^T \\
&= \frac{n^2}{n^2-1} U^T (I - \frac{2}{n+1} e_1 e_1^T) U \\
&= \frac{n^2}{n^2-1} (I - \frac{2}{n+1} (U^T e_1)(e_1^T U)) \\
&= \frac{n^2}{n^2-1} (I - \frac{2}{n+1} (-c)(-c^T)) \\
&= \frac{n^2}{n^2-1} (I - \frac{2}{n+1} c c^T).
\end{aligned}$$

Therefore in this case,

$$E' = \{x \in \mathbb{R}^n : (x - \hat{a})^T \hat{A}^{-1} (x - \hat{a}) \le 1\}.$$

Since we only performed a rotation, the volume did not change. So volume$(E') \le e^{-\frac{1}{2(n+1)}}$volume$(E_0)$.

1.5 General Case: (not covered in lecture on Oct. 31 2014)

Now what if E is not the unit sphere but a general ellipsoid? The idea is to transform E into unit sphere centered at origin via transform $T(x) = y$, apply the result of the previous case, then transform it back via T^{-1}.

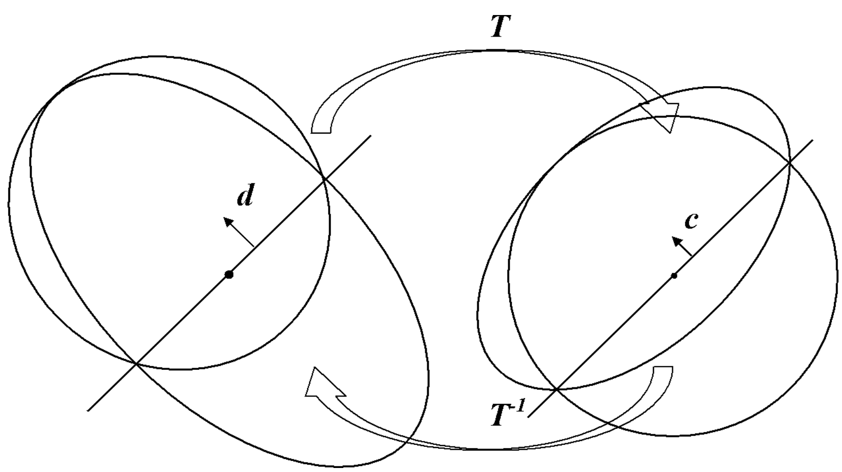


Figure 4: *Case of General Ellipsoid*

Let $E = E_k = E(a_k, A_k)$. Since A_k is positive definite, $A_k = B^T B$ for some B. Then $A_k^{-1} = B^{-1}(B^{-1})^T$, and

$$E(a_k, A_k) = \{x : (x - a_k)^T B^{-1} (B^{-1})^T (x - a_k) \le 1\}.$$

If we set $y = T(x) = (B^{-1})^T(x - a_k)$, we will get

$$y^T y \le 1.$$

So T transforms E_k into $E(0, I)$. $T^{-1}(y) = x = B^T y + a_k$.

The hyperplane in the original space $d^T x \le d^T a_k$ becomes $d^T(B^T y + a_k) \le d^T a_k$, thus $d^T B^T y \le 0$ after the transform T. We want $c^T y \le 0$ for $\|c\| = 1$, therefore set

$$c^T = \frac{d^T B^T}{\|d^T B^T\|},$$

hence

$$c = \frac{Bd}{\sqrt{d^T A d}}.$$

In the transformed space, we have

$$E' = \left\{ y : (y + \frac{1}{n+1}c)^T F^{-1}(y + \frac{1}{n+1}c) \le 1 \right\},$$

where

$$F = \hat{A} = \frac{n^2}{n^2-1}(I - \frac{2}{n+1}cc^T).$$

Now substitute $y = (B^{-1})^T(x - a_k)$ to get back to the original space. We have

$$E_{k+1} = \left\{ x : \left((B^{-1})^T(x - a_k) + \frac{1}{n+1}c \right)^T F^{-1} \left((B^{-1})^T(x - a_k) + \frac{1}{n+1}c \right) \le 1 \right\},$$

$$E_{k+1} = \left\{ x : \left((x - a_k)^T B^{-1} + \frac{1}{n+1}c^T \right) F^{-1} \left((B^{-1})^T(x - a_k) + \frac{1}{n+1}c \right) \le 1 \right\}.$$

If we set $a_{k+1} = a_k - \frac{1}{n+1}B^T c$, then

$$E_{k+1} = \{x : (x - a_{k+1})^T B^{-1} F^{-1} (B^{-1})^T (x - a_{k+1}) \le 1\}.$$

If we set $\hat{F}^{-1} = B^{-1}F^{-1}(B^{-1})^T$, then

$$\begin{aligned}
\hat{F} = B^T F B &= \frac{n^2}{n^2-1} B^T \left(I - \frac{2}{n-1}cc^T \right) B \\
&= \frac{n^2}{n^2-1} \left(A_k - \frac{2}{n+1}(B^T c)(B^T c)^T \right) \\
&= \frac{n^2}{n^2-1} \left(A_k - \frac{2}{n+1}bb^T \right),
\end{aligned}$$

where we set $b = B^T c$. Then $a_{k+1} = a_k - \frac{b}{n+1}$, and $A_{k+1} = \hat{F} = \frac{n^2}{n^2-1}\left(A_k - \frac{2}{n+1}bb^T \right)$.

Since the ratios of volumes are preserved under linear transformation,

$$\frac{\text{volume}(E_{k+1})}{\text{volume}(E_k)} = \frac{\text{volume}(E')}{\text{volume}(E_0)} \le e^{-\frac{1}{2(n+1)}}.$$