1 Strong duality

Recall the two versions of Farkas’ Lemma proved in the last lecture:

**Theorem 1 (Farkas’ Lemma)** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two conditions holds:

1. $\exists x \in \mathbb{R}^n$ such that $Ax = b$, $x \geq 0$;
2. $\exists y \in \mathbb{R}^m$ such that $A^T y \geq 0$, $y^T b < 0$.

**Theorem 2 (Farkas’ Lemma’)** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two conditions holds:

1'. $\exists x \in \mathbb{R}^n$ such that $Ax \leq b$;
2'. $\exists y \in \mathbb{R}^m$ such that $A^T y = 0$, $y^T b < 0$, $y \geq 0$.

The following condition is equivalent to (2'):

2''. $\exists y \in \mathbb{R}^m$ such that $A^T y = 0$, $y^T b = -1$, $y \geq 0$.

These results lead to strong duality, which we will prove in the context of the following primal-dual pair of LPs:

\[
\begin{align*}
\max \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b \\
\min \quad & b^T y \\
\text{s.t.} \quad & A^T y = c \\
& y \geq 0
\end{align*}
\]  

(1)

**Theorem 3 (Strong Duality)** There are four possibilities:

1. Both primal and dual have no feasible solutions (are infeasible).
2. The primal is infeasible and the dual unbounded.
3. The dual is infeasible and the primal unbounded.
4. Both primal and dual have feasible solutions and their values are equal.

**Proof:** There are four possible cases:
Case 1: Infeasible primal, infeasible dual.

We showed in problem 1 of the second homework that it is possible for both the primal and dual to be infeasible.

Case 2: Infeasible primal, feasible dual.

Let \( \bar{y} \) be a feasible solution for the dual and assume the primal is infeasible. Condition \((1)'\) of Farkas’ Lemma’ does not hold, so \((2)'\) must hold, i.e. there exists \( \hat{y} \) such that \( A^T \hat{y} = 0 \), \( \hat{y}^T b < 0 \), and \( \hat{y} \geq 0 \). Consider the family of solutions \( y = \bar{y} + \lambda \hat{y}, \lambda \geq 0 \). For each \( \lambda \), \( y \) is dual-feasible since

\[
A^T y = A^T(\bar{y} + \lambda \hat{y}) = c + \lambda \cdot 0 = c
\]

and

\[
y = \bar{y} + \lambda \hat{y} \geq 0.
\]

The objective value of \( y \) is

\[
y^T b = (\bar{y} + \lambda \hat{y})^T b = \hat{y}^T b + \lambda \hat{y}^T b.
\]

Since \( \hat{y}^T b < 0 \), \( \lim_{\lambda \to \infty} y^T b = -\infty \). Thus, if the primal is infeasible, then the dual is unbounded.

Case 3: Infeasible dual, feasible primal.

Let \( \bar{x} \) be a feasible solution for the primal and assume the dual is infeasible, so that there does not exist \( y \) such that \( A^T y = c, y \geq 0 \). Condition \((1)\) of the original Farkas’ Lemma (with renamed symbols \( A \to A^T, x \to y, b \to c \)) does not hold, so \((2)\) must hold, i.e. there exists an \( \hat{x} \) such that \( A\hat{x} \geq 0, c^T \hat{x} < 0 \). Consider the solution \( x = \bar{x} - \lambda \hat{x} \) for \( \lambda \geq 0 \). For each \( \lambda \), \( x \) is primal-feasible:

\[
Ax = A(\bar{x} - \lambda \hat{x}) \leq b - \lambda A\hat{x} \leq b,
\]

The objective value of \( x \) is

\[
c^T x = c^T(\bar{x} - \lambda \hat{x}) = c^T \bar{x} - \lambda c^T \hat{x}.
\]

Since \( c^T \hat{x} < 0 \), \( \lim_{\lambda \to \infty} c^T x = +\infty \). Thus, if the dual is infeasible, then the primal is unbounded.

Case 4: Feasible primal, feasible dual.

Let \( \bar{x} \) and \( \bar{y} \) be feasible solutions to the primal and dual, respectively. By weak duality, \( c^T \bar{x} \leq \bar{y}^T b \), so both the primal and dual are bounded. Let \( \gamma \) be the optimal value of the dual. Suppose that the optimal value of the primal were less than \( \gamma \), that is, suppose \( \exists x \) satisfying

\[
Ax \leq b, \quad c^T x \geq \gamma \iff \begin{bmatrix} A \\ -c^T \end{bmatrix} x \leq \begin{bmatrix} b \\ -\gamma \end{bmatrix}
\]

This is equivalent to the statement that condition \((1)'\) of Farkas’ Lemma’ does not hold, so condition \((2)'\) must hold. Thus, there exists a vector (call it \( [y^T, \lambda]^T \), where \( \lambda \in \mathbb{R} \)) satisfying

\[
\begin{bmatrix} A \\ -c^T \end{bmatrix}^T \begin{bmatrix} y \\ \lambda \end{bmatrix} = 0, \quad \begin{bmatrix} b \\ -\gamma \end{bmatrix}^T \begin{bmatrix} y \\ \lambda \end{bmatrix} < 0, \quad \begin{bmatrix} y \\ \lambda \end{bmatrix} \geq 0
\]

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We would like to divide by $\lambda \in \mathbb{R}$, which requires showing that $\lambda \neq 0$. To see this, suppose $\lambda = 0$. This implies that $A^T y = 0$, $b^T y < 0$, and $y \geq 0$, meaning condition (2$'$) of Farkas' Lemma holds. Therefore, condition (1$'$) must not hold, i.e. there does not exist $x$ such that $Ax \leq b$. This contradicts the assumption of primal feasibility, however, so $\lambda > 0$.

The vector $\left( \frac{y}{\lambda} \right)$ is dual-feasible, because $\left( \frac{y}{\lambda} \right) \geq 0$ and

$$A^T y - \lambda c = 0 \implies A^T \left( \frac{y}{\lambda} \right) = c.$$ 

However, $b^T y - \lambda \gamma < 0$, so $b^T \left( \frac{y}{\lambda} \right) < \gamma$, which contradicts the assumption that $\gamma$ is the optimal value of the dual. Thus, if the primal and dual are both feasible, then their optimal values are equal.

\[\square\]

2 Optimality conditions

Consider the primal-dual pair of LPs in standard form:

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \quad b^T y \\
\text{s.t.} & \quad A^T y \leq c
\end{align*}
\]

Given a primal-feasible $x$, how can one tell whether $x$ is optimal?

Answer 1 By strong duality, $x$ is optimal if there exists a dual-feasible $y$ such that $c^T x = b^T y$.

This is true as far as it goes, but it doesn’t seem that useful. Let’s think about other ways in which we can show the optimality of $x$.

Let $x$ and $y$ be feasible for the primal and dual, respectively. Recall the proof of weak duality:

\[
c^T x = \sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} y_i \left( \sum_{j=1}^{n} a_{ij} x_j \right) = \sum_{i=1}^{m} y_i b_i = b^T y
\]

where the inequality follows from $A^T y \leq c$. By strong duality, if $x$ and $y$ are optimal, then $c^T x = b^T y$, i.e. each of the $n$ inequalities above must be binding. This occurs iff, for all $j \in \{1, \ldots, n\}$, either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$. Call these conditions (*).

Definition 1 We say that a primal-feasible $x$ and a dual-feasible $y$ obey the complementary slackness conditions if (* holds).

So we see from the above that if $x$ and $y$ are optimal solutions, then complementary slackness holds. But actually we can say something stronger than this.
Lemma 4 Given a primal-feasible solution \( x \) and a dual-feasible solution \( y \), \( x \) and \( y \) are optimal if and only if the complementary slackness conditions hold.

Hence we have another answer to our question.

Answer 2 \( x \) is optimal if there exists a dual-feasible \( y \) such that the complementary slackness conditions hold.

This still doesn’t seem like such a useful way of verifying optimality, but it will prove to be a step in the right direction.

So far we haven’t been taking advantage of something that we know about optimal solutions, namely that there exists an optimal solution that is a vertex. We’ve also shown in a problem set that if \( x \) is not a vertex, we can find a vertex \( \tilde{x} \) such that \( c^T \tilde{x} \leq c^T x \). So we can assume that \( x \) is a vertex.

Recall that \( x \) is a vertex if and only if \( \text{rank}(A_x) = n \). Note that \( m \) inequalities are necessarily binding, since \( a_j^T x = b_j \) for all \( j \in \{1, \ldots, m\} \), where \( a_j^T \) is the \( j \)th row of \( A \). The remaining \( n - m \) binding inequalities (modulo linear dependence) must be of the form \( x_i = 0 \). Assume that the variables are numbered such that \( x_1, \ldots, x_k > 0 \) and \( x_{k+1}, \ldots, x_n = 0 \). Then

\[
\begin{bmatrix}
A \\
0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
1 \\
\end{bmatrix} =
\begin{bmatrix}
b \\
0 \\
\end{bmatrix}.
\]

The matrix \( A^- \in \mathbb{R}^{(m+n-k) \times n} \) has rank \( n \), so all its columns are linearly independent. Therefore, the columns of \( A \) corresponding to positive \( x_i \) variables must also be linearly independent. This gives us the following lemma.

Lemma 5 A feasible solution \( x \) is a vertex iff the columns of \( A \) corresponding to the positive components of \( x \) are linearly independent.

Definition 2 A set \( B \) of \( m \) columns of \( A \) is a basis if these columns are linearly independent.