ORIE 6300 Mathematical Programming I September 18, 2014

Lecture 8

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1 Strong duality

Recall the two versions of Farkas' Lemma proved in the last lecture:

Theorem 1 (Farkas' Lemma) *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two condition holds:*

(1) $\exists\, x \in \mathbb{R}^n$ *such that* $Ax = b$, $x \geq 0$*;*

(2) $\exists\, y \in \mathbb{R}^m$ *such that* $A^T y \geq 0$, $y^T b < 0$.

Theorem 2 (Farkas' Lemma′) *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two condition holds:*

(1′) $\exists\, x \in \mathbb{R}^n$ *such that* $Ax \leq b$*;*

(2′) $\exists\, y \in \mathbb{R}^m$ *such that* $A^T y = 0$, $y^T b < 0$, $y \geq 0$.

The following condition is equivalent to (2′)*:*

(2″) $\exists\, y \in \mathbb{R}^m$ *such that* $A^T y = 0$, $y^T b = -1$, $y \geq 0$.

These results lead to strong duality, which we will prove in the context of the following primal-dual pair of LPs:

$$\begin{array}{rlcrl} \max & c^T x & \qquad & \min & b^T y \\ \text{s.t.} & Ax \leq b & & \text{s.t.} & A^T y = c \\ & & & & y \geq 0 \end{array} \tag{1}$$

Theorem 3 (Strong Duality) *There are four possibilities:*

1. *Both primal and dual have no feasible solutions (are* infeasible*).*
2. *The primal is infeasible and the dual unbounded.*
3. *The dual is infeasible and the primal unbounded.*
4. *Both primal and dual have feasible solutions and their values are equal.*

Proof: There are four possible cases:

Case 1: Infeasible primal, infeasible dual.

We showed in problem 1 of the second homework that it is possible for both the primal and dual to be infeasible.

Case 2: Infeasible primal, feasible dual.

Let $\bar{y}$ be a feasible solution for the dual and assume the primal is infeasible. Condition $(1')$ of Farkas' Lemma$'$ does not hold, so $(2')$ must hold, *i.e.* there exists $\hat{y}$ such that $A^T\hat{y} = 0$, $\hat{y}^T b < 0$, and $\hat{y} \geq 0$. Consider the family of solutions $y = \bar{y} + \lambda\hat{y}$, $\lambda \geq 0$. For each λ, y is dual-feasible since

$$A^T y = A^T(\bar{y} + \lambda\hat{y}) = c + \lambda \cdot 0 = c$$

and

$$y = \bar{y} + \lambda\hat{y} \geq 0.$$

The objective value of y is

$$y^T b = (\bar{y} + \lambda\hat{y})^T b = \bar{y}^T b + \lambda\hat{y}^T b.$$

Since $\hat{y}^T b < 0$, $\lim_{\lambda\to\infty} y^T b = -\infty$. Thus, if the primal is infeasible, then the dual is unbounded.

Case 3: Infeasible dual, feasible primal.

Let $\bar{x}$ be a feasible solution for the primal and assume the dual is infeasible, so that there does not exist y such that $A^T y = c$, $y \geq 0$. Condition (1) of the original Farkas' Lemma (with renamed symbols $A \to A^T$, $x \to y$, $b \to c$) does not hold, so (2) must hold, *i.e.* there exists an $\hat{x}$ such that $A\hat{x} \geq 0$, $c^T\hat{x} < 0$. Consider the solution $x = \bar{x} - \lambda\hat{x}$ for $\lambda \geq 0$. For each λ, x is primal-feasible:

$$Ax = A(\bar{x} - \lambda\hat{x}) \leq b - \lambda A\hat{x} \leq b,$$

The objective value of x is

$$c^T x = c^T(\bar{x} - \lambda\hat{x}) = c^T\bar{x} - \lambda c^T\hat{x}.$$

Since $c^T\hat{x} < 0$, $\lim_{\lambda\to\infty} c^T x = +\infty$. Thus, if the dual is infeasible, then the primal is unbounded.

Case 4: Feasible primal, feasible dual.

Let $\bar{x}$ and $\bar{y}$ be feasible solutions to the primal and dual, respectively. By weak duality, $c^T\bar{x} \leq \bar{y}^T b$, so both the primal and dual are bounded. Let γ be the optimal value of the dual. Suppose that the optimal value of the primal were less than γ, that is, suppose $\nexists x$ satisfying

$$Ax \leq b,\ c^T x \geq \gamma \quad \Longleftrightarrow \quad \begin{bmatrix} A \\ -c^T \end{bmatrix} x \leq \begin{bmatrix} b \\ -\gamma \end{bmatrix}$$

This is equivalent to the statement that condition $(1')$ of Farkas' Lemma$'$ does not hold, so condition $(2')$ must hold. Thus, there exists a vector (call it $[y^T, \lambda]^T$, where $\lambda \in \mathbb{R}$) satisfying

$$\begin{bmatrix} A \\ -c^T \end{bmatrix}^T \begin{bmatrix} y \\ \lambda \end{bmatrix} = 0, \quad \begin{bmatrix} b \\ -\gamma \end{bmatrix}^T \begin{bmatrix} y \\ \lambda \end{bmatrix} < 0, \quad \begin{bmatrix} y \\ \lambda \end{bmatrix} \geq 0$$

We would like to divide by $\lambda \in \mathbb{R}$, which requires showing that $\lambda \neq 0$. To see this, suppose $\lambda = 0$. This implies that $A^T y = 0$, $b^T y < 0$, and $y \geq 0$, meaning condition $(2')$ of Farkas' Lemma$'$ holds.Therefore, condition $(1')$ must not hold,*i.e.* there does not exist x such that $Ax \leq b$. This contradicts the assumption of primal feasibility, however, so $\lambda > 0$.

The vector $\left(\frac{y}{\lambda}\right)$ is dual-feasible, because $\left(\frac{y}{\lambda}\right) \geq 0$ and

$$A^T y - \lambda c = 0 \quad \Rightarrow \quad A^T \left(\frac{y}{\lambda}\right) = c.$$

However, $b^T y - \lambda\gamma < 0$, so $b^T \left(\frac{y}{\lambda}\right) < \gamma$, which contradicts the assumption that γ is the optimal value of the dual. Thus, if the primal and dual are both feasible, then their optimal values are equal.

□

2 Optimality conditions

Consider the primal-dual pair of LPs in standard form:

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & \end{array} \tag{2}$$

Given a primal-feasible x, how can one tell whether x is optimal?

Answer 1 *By strong duality, x is optimal if there exists a dual-feasible y such that $c^T x = b^T y$.*

This is true as far as it goes, but it doesn't seem that useful. Let's think about other ways in which we can show the optimality of x.

Let x and y be feasible for the primal and dual, respectively. Recall the proof of weak duality:

$$\begin{aligned} c^T x = \sum_{j=1}^{n} c_j x_j &\geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j \\ &= \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) = \sum_{i=1}^{m} y_i b_i = b^T y \end{aligned}$$

where the inequality follows from $A^T y \leq c$. By strong duality, if x and y are optimal, then $c^T x = b^T y$, *i.e.* each of the n inequalities above must be binding. This occurs iff, for all $j \in \{1, \ldots, n\}$, either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$. Call these conditions $(*)$.

Definition 1 *We say that a primal-feasible x and a dual-feasible y obey the* complementary slackness conditions *if $(*)$ holds.*

So we see from the above that if x and y are optimal solutions, then complementary slackness holds. But actually we can say something stronger than this.

Lemma 4 *Given a primal-feasible solution x and a dual-feasible solution y, x and y are optimal if and only if the complementary slackness conditions hold.*

Hence we have another answer to our question.

Answer 2 *x is optimal if there exists a dual-feasible y such that the complementary slackness conditions hold.*

This still doesn't seem like such a useful way of verifying optimality, but it will prove to be a step in the right direction.

So far we haven't been taking advantage of something that we know about optimal solutions, namely that there exists an optimal solution that is a vertex. We've also shown in a problem set that if x is not a vertex, we can find a vertex $\tilde{x}$ such that $c^T\tilde{x} \leq c^T x$. So we can assume that x is a vertex.

Recall that x is a vertex if and only if $rank(A_=) = n$. Note that m inequalities are necessarily binding, since $a_j^T x = b_j$ for all $j \in \{1, \ldots, m\}$, where a_j^T is the j^{th} row of A. The remaining $n - m$ binding inequalities (modulo linear dependence) must be of the form $x_i = 0$. Assume that the variables are numbered such that $x_1, ..., x_k > 0$ and $x_{k+1}, ..., x_n = 0$. Then

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{A} \\ \hline 0 & I \end{array}\right] x = \left[\begin{array}{c} b \\ \hline 0 \end{array}\right].$$

The matrix $A^{=} \in \mathbb{R}^{(m+n-k)\times n}$ has rank n, so all its columns are linearly independent. Therefore, the columns of A corresponding to positive x_i variables must also be linearly independent. This gives us the following lemma.

Lemma 5 *A feasible solution x is a vertex iff the columns of A corresponding to the positive components of x are linearly independent.*

Definition 2 *A set B of m columns of A is a* basis *if these columns are linearly independent.*