Lecture 7

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1 Review

A while back, we defined polyhedrons and polytopes as follows.

Definition 1 *A* Polyhedron *is* $P = \{x \in \Re^n : Ax \leq b\}$

Definition 2 *A* Polytope *is given by* $Q = conv(v_1, v_2, ..., v_k)$*, where the* v_i *are the vertices of the polytope, for* k *finite.*

Also recall the equivalence of extreme points, vertices and basic feasible solutions, and recall the definition of a bounded polyhedron.

Definition 3 *A polyhedron* P *is* bounded *iff* $\exists M > 0$ *such that* $||x|| \leq M \forall x \in P$*.*

We showed bounded polyhedra were polytopes by taking the extreme points and seeing that they were the verticies for P as a polytope.

Recall also the Separating Hyperplane Theorem from a previous lecture.

Theorem 1 *(Separating Hyperplane) Let* $C \subseteq \Re^n$ *be a closed, nonempty and convex set. Let* $y \in \Re^n, y \notin C$*. Then there exists* $0 \neq a \in \Re^n, b \in \Re$ *such that* $a^T y > b$ *and* $a^T x < b$ *for all* $x \in C$*.*

2 The polar of a set

Now we want to prove that polytopes are bounded polyhedra. To do this, we need to introduce one more concept.

Definition 4 *If* $S \subseteq \Re^n$*, then its* polar *is* $S^\circ = \{z \in \Re^n : z^T x \leq 1, \forall x \in S\}$*.*

Lemma 2 *If* C *is a closed convex subset of* $\Re^n$ *with* $0 \in C$*, then* $C^{\circ\circ} := (C^\circ)^\circ = C$*.*

Proof:

- ($\supseteq$) If $x \in C$, we want to show that $x \in C^{\circ\circ}$, i.e., that $z^T x \leq 1$ for all $z \in C^\circ$. But $z \in C^\circ$ implies $z^T x \leq 1$, so this holds.

- ($\subseteq$) We will show that if $x \notin C$, then $x \notin C^{\circ\circ}$. First note that C is closed and convex with atleast $z = 0 \in C$. If $x \notin C$, then by the Separating Hyperplane Theorem, there exists $0 \neq a \in \Re^n$ and $b \in \Re$ with $a^T x > b > a^T z$ for all $z \in C$. Since $0 \in C$, then $b > 0$. Let $\tilde{a} = a/b \neq 0$. Therefore $\tilde{a}^T x > 1 > \tilde{a}^T z$, for all $z \in C$. This implies $\tilde{a} \in C^\circ$. But $\tilde{a}^T x > 1$, so $x \notin C^{\circ\circ}$.

Therefore $C^{\circ\circ} = C$. □

3 Polytopes are Bounded Polyhedra

Now we can prove our result, at least sort of. We'll assume that 0 is in the interior of the polytope. We claim that this can be done without loss of generality; this is because we can translate the polytope to have $0 \in P$, apply the following proof and then translate back if needed.

Theorem 3 *If $Q \subseteq \Re^n$ is a polytope with 0 in the interior of Q, then Q is a (bounded) polyhedron.*

Proof: Our proof strategy is as follows. We will first show that the polar of a polytope is a polyhedron. We then show that that since the polytope has 0 in its interior, then the polar of the polytope is bounded. So then $P = Q^\circ$ is a bounded polyhedron. We know from a previous lecture that any bounded polyhedron is a polytope, so $P = Q^\circ$ is a polytope. But then applying the proof that the polar of a polytope is a polyhedron, we get that $P^\circ = Q^{\circ\circ} = Q$ (by the lemma above) is a polyhedron. It is easy to prove that a polytope is bounded.

We first prove that the polar of Q is a polyhedron. Let $P = Q^\circ$. Then we know that $P^\circ = Q^{\circ\circ} = Q$. Since Q is a polytope, $Q = \text{conv}\{v_1, \ldots, v_k\}$ for some k finite vectors $v_1, \ldots, v_k \in \Re^n$. Now $P = Q^\circ = \{z \in \Re^n : x^T z \leq 1, \forall x \in Q\}$, so $v_i^T z = z^T v_i \leq 1$ for $i = 1, 2, \ldots, k$. For any $x \in Q, x = \sum_{i=1}^k \lambda_i v_i$ where $\lambda_i \geq 0, \sum_i \lambda_i = 1$. Therefore if $z^T v_i \leq 1$ for $i = 1, \ldots, k$,

$$z^T x = z^T (\sum_{i=1}^k \lambda_i v_i) = \sum_{i=1}^k \lambda_i (z^T v_i) \leq \sum_{i=1}^k \lambda_i = 1.$$

Therefore

$$P = \{z \in \Re^n : v_i^T z \leq 1,\, i = 1, \ldots, k\},$$

so P is a polyhedron.

Now we need to show that the fact that Q has 0 in its interior implies Q° is bounded. $0 \in int(Q) \Rightarrow \exists$ some $\epsilon > 0$, all $x \in \Re^n$ with $||x|| \leq \epsilon$ lie in Q. If $z \in P$, $z \neq 0$, then

$$x = \epsilon \frac{z}{||z||} \in Q.$$

since $||x|| \leq \epsilon$. Then since $P = Q^\circ$,

$$x^T z \leq 1 \quad \Rightarrow \quad \frac{\epsilon z^T z}{||z||} \leq 1 \quad \Rightarrow \quad ||z|| \leq \frac{1}{\epsilon} \equiv M,$$

where M is the bound. Hence P is a bounded polyhedron, and from the sketch at the beginning of the proof we get that Q is a polyhedron. □

4 Farkas' Lemma

We are now finally almost able to prove strong duality. We will first need to show two lemmas before we are able to do this. On a side note, Farkas means "wolf" in Hungarian. Just some trivia.

Theorem 4 (Farkas' Lemma) *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then **exactly** one of the following two condition holds for a given A, b:*

(1) $\exists\, x \in \mathbb{R}^n$ *such that* $Ax = b,\ x \geq 0$;

(2) $\exists\, y \in \mathbb{R}^m$ *such that* $A^T y \geq 0,\ y^T b < 0$.

Proof: First we show that we can't have both (1) and (2). Assume for contradiction $\exists \hat{x}$ such that $A\hat{x} = b, \hat{x} \geq 0$, and $\exists \hat{y}$ such that $A^T\hat{y} \geq 0$, $y^T b \leq 0$. Note that $\hat{y}^T A\hat{x} = \hat{y}^T(A\hat{x}) = \hat{y}^T b < 0$ since by (1), $A\hat{x} = b$ and by (2) $\hat{y}^T b < 0$. But also $\hat{y}^T A\hat{x} = (\hat{y}^T A)\hat{x} = (A^T\hat{y})^T\hat{x} \geq 0$ since by (2) $A^T\hat{y} \geq 0$ and by (1) $\hat{x} \geq 0$.

Now we must show that if (1) doesn't hold, then (2) does. To do this, let $v_1, v_2, \ldots, v_n$ be the columns of A. Define

$$Q = cone(v_1, \ldots, v_n) \equiv \{s \in \Re^m : s = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \forall i\}.$$

This is a conic combination of the columns of A, which differs from a convex combination since we don't require that $\sum_{i=1}^n \lambda_i = 1$. Then $Ax = \sum_{i=1}^n x_i v_i$, there exists an x such that $Ax = b$ and $x \geq 0$ if and only if $b \in Q$ as x's are weights λ_i.

So if (1) does not hold then $b \notin Q$. We show that condition (2) must hold. We know that Q is nonempty (since $0 \in Q$), closed, and convex, so we can apply the separating hyperplane theorem. The theorem implies that there exists $\alpha \in \Re^m$, $\alpha \neq 0$, and β such that $\alpha^T b > \beta$ and $\alpha^T s < \beta$ for all $s \in Q$. Since $0 \in Q$, we know that $\beta > 0$. Note also that $\lambda v_i \in Q$ for all $\lambda > 0$. Then since $\alpha^T s < \beta$ for all $s \in Q$, we have $\alpha^T(\lambda v_i) \in Q$ for all $\lambda > 0$, so that $\alpha^T v_i < \beta/\lambda$ for all $\lambda > 0$. Since $\beta > 0$, as $\lambda \to \infty$, we have that $\alpha^T v_i \leq 0$. Thus by setting $y = -\alpha$, we obtain $y^T b < 0$ and $y^T v_i \geq 0$ for all i. Since the v_i are the columns of A, we get that $A^T y \geq 0$. Thus condition (2) holds. $\square$

Now will show the equivalence of a variant on Farkas' Lemma.

Theorem 5 (Farkas' Lemma′) *Let* $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$. *Then exactly one of the following two condition holds:*

(1′) $\exists\, x \in \mathbb{R}^n$ *such that* $Ax \leq b$;

(2′) $\exists\, y \in \mathbb{R}^m$ *such that* $A^T y = 0,\ y^T b < 0,\ y \geq 0$.

The following condition is equivalent to (2′):

(2″) $\exists\, y \in \mathbb{R}^m$ *such that* $yA = 0,\ y^T b = -1,\ y \geq 0$.

Proof: First we prove that (2′) if and only if (2″). Clearly if (2″) is true, then (2′) is true. If (2′) is true, let $\hat{y} = -\frac{1}{y^T b} y$. Then $\hat{y} \geq 0$ since $y \geq 0$ and $y^T b < 0$. Also

$$\hat{y}^T b = -\frac{y^T b}{y^T b} = -1,$$

and

$$A^T\hat{y} = \frac{-1}{y^T b}(A^T y) = 0,$$

where the last equation follows from $A^T y = 0$ in (2″).

As before, we cannot have both (1′) and (2′). Suppose otherwise. Then $\exists x$ such that $Ax \leq b$ and $\exists y$ such that $A^T y = 0$ and $y^T b < 0$. Then as before $y^T Ax = y^T(Ax) \leq y^T b < 0$, since $Ax = b$

and $y^T b < 0$, and also $y^T A x = (y^T A)x = (A^T y)^T x = 0$, since $A^T y = 0$. This gives the desired contradiction.

Now suppose (2′) does not hold, so (2″) does not hold either; want to show (1′) holds. Rewrite the system $A^T y = 0$, $y^T b = -1$ as:

$$\bar{A} = \begin{bmatrix} A^T \\ b^T \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}.$$

Then since (2″) does not hold, there does not exist $z \in \Re^m$ such that $z \geq 0$ and $\bar{A}z = \bar{b}$. This is just a rewriting of condition (1) of the original Farkas' Lemma such that (1) does not hold. Therefore condition (2) must hold, which implies that there exists s such that $\bar{A}^T s \geq 0$ and $\bar{b}^T s < 0$. Set

$$s = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

for $x \in \Re^n$ and $\lambda \in \Re$. Then $\bar{b}^T s < 0$ implies that

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} < 0,$$

which implies that $\lambda > 0$. Also, $\bar{A}^T s \geq 0$ implies that

$$\begin{bmatrix} A^T \\ b^T \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0,$$

which implies that

$$\begin{bmatrix} A & b \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0,$$

or that $Ax + \lambda b \geq 0$, or that $Ax \geq -\lambda b$, or that $A(\frac{-x}{\lambda}) \leq b$. Therefore $-x/\lambda$ satisfies (1′) and implies it's true. This concludes our proofs of the Farkas' Lemmas. □