The Two-Stripe Symmetric Circulant TSP is in P

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Abstract

The symmetric circulant TSP is a special case of the traveling salesman problem in which edge costs are symmetric and obey circulant symmetry. Despite the substantial symmetry of the input, remarkably little is known about the symmetric circulant TSP. The complexity of the problem has been an often-cited open question. Considerable effort has been made to understand the case in which only edges of two lengths a_1 and a_2 are allowed to have finite cost: the two-stripe symmetric circulant TSP (see Greco and Gerace [10] and Gerace and Greco [8]). In this paper, we resolve the complexity of the two-stripe symmetric circulant TSP, providing the first step toward resolving the polynomial-time solvability of circulant TSP. To do so, we reduce two-stripe symmetric circulant TSP to the problem of finding certain minimum-cost Hamiltonian paths on *cylindrical graphs*. We then solve this Hamiltonian path problem. Our results show that the two-stripe symmetric circulant TSP is in P. Note that the input size of a two-stripe symmetric circulant TSP instance is a constant number of numbers (including n, the number of cities), so that a polynomial-time algorithm for the decision problem must run in time polylogarithmic in n, and a polynomial-time algorithm for the optimization problem cannot output the tour. We address this latter difficulty by showing that the optimal tour must fall into one of two parameterized classes of tours, and that we can output the class and the parameters in polynomial time.

1 Introduction

The traveling salesman problem (TSP) is one of the most famous problems in combinatorial optimization. An input to the TSP consists of a set of n cities $[n] := \{1, 2, ..., n\}$ and edge costs c_{ij} for each pair of distinct $i, j \in [n]$ representing the cost of traveling from city i to city j. Given this information, the TSP is to find a minimum-cost tour visiting every city exactly once. Throughout this paper, we implicitly assume that the edge costs are *symmetric* (so that $c_{ij} = c_{ji}$ for all distinct $i, j \in [n]$) and interpret the n cities as vertices of the complete undirected graph K_n with edge costs $c_e = c_{ij}$ for edge $e = \{i, j\}$. In this setting, the TSP is to find a minimum-cost Hamiltonian cycle on K_n.

With just this set-up, the TSP is well known to be NP-hard. An algorithm that could approximate TSP solutions in polynomial time to within any constant factor α would imply P=NP (see, e.g., Theorem 2.9 in Williamson and Shmoys [21]). Thus it is common to consider special cases, such as requiring costs to obey the triangle inequality (i.e. requiring costs to be *metric*, so that $c_{ij} + c_{jk} \geq c_{ik}$ for all $i, j, k \in [n]$), or costs that are distances between points in Euclidean space. Another special case that has been considered is the $(1, 2)$-TSP which restricts $c_{ij} \in \{1, 2\}$ for every edge $\{i, j\}$ (see, e.g., Papadimitriou and Yannakakis [18], Berman and Karpinski [1], Karpinski and Schmied [15]).

In this paper, we consider a different class of instances: circulant TSP. This class can be described by **circulant matrices,** matrices of the form

$$\begin{pmatrix} m_0 & m_1 & m_2 & m_3 & \cdots & m_{n-1} \\ m_{n-1} & m_0 & m_1 & m_2 & \cdots & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & m_1 & \ddots & m_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & m_4 & \cdots & m_0 \end{pmatrix} = \left(m_{(t-s) \bmod n}\right)_{s,t=1}^{n}. \tag{1}$$

In **circulant TSP**, the matrix of edge costs $C = (c_{ij})_{i,j=1}^n$ is circulant; the cost of edge $\{i, j\}$ only depends on $i - j \bmod n$. Our assumption that the edge costs are symmetric and that K_n is a simple graph implies that, for symmetric circulant TSP instances, we can write our cost matrix in terms of $\lfloor \frac{n}{2} \rfloor$ parameters:

$$C = (c_{(j-i) \bmod n})_{i,j=1}^n = \begin{pmatrix} 0 & c_1 & c_2 & c_3 & \cdots & c_1 \\ c_1 & 0 & c_1 & c_2 & \cdots & c_2 \\ c_2 & c_1 & 0 & c_1 & \ddots & c_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & c_4 & \cdots & 0 \end{pmatrix}, \tag{2}$$

with $c_0 = 0$ and $c_i = c_{n-i}$ for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$. See, e.g., Figure 1 for an picture of (symmetric) circulant symmetry. Importantly, in circulant TSP we do not implicitly assume that the edge costs are metric. The original motivations for circulant TSP stem from minimizing wallpaper waste (Garfinkel [6]) and reconfigurable network design (Medova [17]).

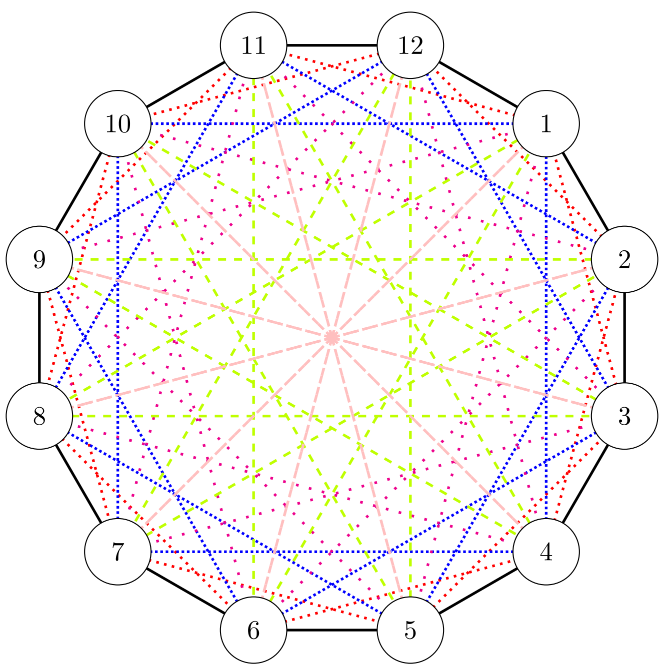


Figure 1: Circulant symmetry. Edges of a fixed length are indistinguishable and have the same cost. E.g. all edges of the form $\{v, v + 1\}$ (where $v + 1$ is taken mod n) have the same appearance.

Circulant TSP is a compelling open problem because of the intriguing structure circulant symmetry broadly provides to combinatorial optimization problems: It seems to provide just enough structure to make an ambiguous set of instances. It is unclear whether or not a given combinatorial optimization problem should remain hard or become easy when restricted to circulant instances. Some classic combinatorial optimization problems are easy when restricted to circulant instances: in the late 70's, Garfinkel [6] considered a restricted set of circulant TSP instances motivated by minimizing wallpaper waste and argued that, for these instances, the canonical greedy algorithm for TSP (the nearest neighbor heuristic) provides an optimal solution. In the late 80's, Burkard and Sandholzer [2] showed that the decidability question for whether or not a symmetric circulant graph (i.e. a graph whose adjacency matrix is circulant) is Hamiltonian can be solved in polynomial time and showed that bottleneck TSP is polynomial-time solvable on symmetric circulant graphs. Bach, Luby, and Goldwasser (cited in Gilmore, Lawler, and Shmoys [9]) showed that one could find minimum-cost Hamiltonian paths in (not-necessarily-symmetric) circulant graphs in polynomial time. In contrast, Codenotti, Gerace, and Vigna [5] show that Max Clique and Graph Coloring remain NP-hard when restricted to circulant graphs and do not admit constant-factor approximation algorithms unless P=NP.

Because of this ambiguity, the complexity of circulant TSP has often been cited as an open problem (see, e.g., Burkhard [3], Burkhard, Deĭneko, Van Dal, Van der Veen, and Woeginger [4], and Lawler, Lenstra, Rinnooy Kan, and Shmoys [16]). It is not known if the circulant TSP is solvable in polynomial-time or is NP-hard, even when restricted to instances where only two of the edge costs $c_1, ..., c_{\lfloor \frac{n}{2} \rfloor}$ are finite: the *symmetric two-stripe circulant TSP (two-stripe TSP)* (see Greco and Gerace [10] and Gerace and Greco [8]). Yang, Burkard, Çela, and Woeginger [22] provide a polynomial-time algorithm for asymmetric TSP in circulant graphs with only two stripes having finite edge costs. The symmetric two-stripe circulant TSP is not, however, a special case of the asymmetric two-stripe version[1].

Despite substantial structure and symmetry, the complexity of two-stripe circulant TSP had previously remained elusive. General upper- and lower-bounds for circulant TSP stem from Van der Veen, Van Dal, and Sierksma [20]; Greco and Gerace [10] and Gerace and Greco [8] focus specifically on the two-stripe TSP, and prove sufficient (but not necessary) conditions for these upper- and lower-bounds to apply. Van der Veen, Van Dal, and Sierksma [20] and Gerace and Greco [7] give a general heuristic for circulant TSP that provides a tour within a factor of two of the optimal solution; no improvements to this general heuristic have been made when constrained to the two-stripe version.

In this paper, we take the first step toward resolving the polynomial-time solvability of circulant TSP by showing that symmetric two-stripe circulant TSP is solvable in polynomial time. We need to be clear on what we mean by "solvable in polynomial time" in this case. The input is the number n (represented in binary by $O(\log n)$ bits), the indices of the two finite cost edges, and the corresponding costs. So to run in polynomial time in this case, we should run in time polylogarithmic in n. We are able to compute the cost of the optimal tour in time polynomial in the input size, and thus the decision problem of whether the cost of the optimal tour is at most a bound given as input is solvable in polynomial time; this places the decision version of the problem in the class P. Notice, however, time polynomial in the input size is not sufficient to output the complete sequence of n vertices to visit in the tour. Nevertheless, given two parameterized classes of tours that we will later describe, we are able to compute in polynomial time to which class the optimal tour belongs, as well as the values of the parameters.

In Section 2, we begin by providing background on circulant TSP. This includes previous work on two-stripe TSP, a useful tool from number theory, and results on the structure of circulant graphs. In Section 3, we introduce cylinder graphs and reduce the two-stripe TSP to finding certain minimum-cost Hamiltonian paths on cylinder graphs. We also introduce a class of Hamiltonian paths, called GG paths after work by Gerace and Greco. We formally state our algorithm in Section 5. Assuming a characterization theorem for Hamiltonian paths between the first and last column in cylinder graphs, we prove the correctness of our algorithm and show that it runs in polynomial time. In Section 6, we prove the aforementioned characterization theorem. We conclude in Section 7.

[1]In the symmetric case, edges $\{v, v+i\}$ and $\{v, v-i\}$ of cost c_i connect v to both $v+i$ and $v-i$; in the asymmetric case, there are edge costs $c_1, \dots, c_{n-1}$ and an edge $(v, v+i)$ of cost c_i only connects from v to $v+i$. To encode two general symmetric circulant edges would require four asymmetric circulant edges.

2 Background and Notation

In this section, we formalize our notation for the two-stripe TSP. We also describe the pertinent background results we will use.

Throughout this paper, we use $\equiv_n$ to denote equivalence mod n. All calculations are implicitly mod n, unless indicated otherwise. For example, we use $v + a_1$ to denote the vertex reachable from v by following an edge of length a_1; the label $v + a_1$ is implicitly taken mod n.

We also frequently rely on the following basic fact about linear congruences. See, e.g., Theorem 57 of Hardy and Wright [14].

Proposition 2.1. *The linear congruence*

$$ax \equiv_n b$$

has a solution if and only if $GCD(a, n)$ *divides* b. *Moreover, there are exactly* $GCD(a, n)$ *solutions which take the form*

$$x_0 + \lambda \frac{n}{GCD(a, n)}, \lambda = 0, 1, ..., GCD(a, n) - 1$$

for some $0 \leq x_0 < \frac{n}{GCD(a,n)}$.

2.1 Circulant Graphs

Recall that a **circulant graph** is a simple graph whose adjacency matrix is circulant. In circulant TSP, all edges $\{i, j\}$ such that $i - j \equiv_n k$ or $i - j \equiv_n (n - k)$ have the same cost c_k. Such edges are typically referred to as in the **k-th stripe**, or as of **length** k. In the two-stripe TSP, only two of the edge costs $c_1, c_2, ..., c_{\lfloor \frac{n}{2} \rfloor}$ are finite. We refer to those two edge lengths as a_1 and a_2, so that $0 \leq c_{a_1} \leq c_{a_2} < \infty$, and for $i \notin \{c_{a_1}, c_{a_2}\}$, $c_i = \infty$.

We use the following definition to describe circulant graphs including exactly the edges associated with some set of stripes S. In the two-stripe TSP, we are generally interested in subsets S of size 1 or 2.

Definition 2.2. *Let* $S \subset \{1, ..., \lfloor \frac{n}{2} \rfloor\}$. *The* **circulant graph** $C\langle S \rangle$ *is the (simple, undirected, unweighted) graph including exactly the edges associated with the stripes* S. *I.e., the graph with adjacency matrix*

$$A = (a_{ij})_{i,j=1}^{n}, \qquad a_{ij} = \begin{cases} 1, & (i - j) \bmod n \in S \text{ or } (j - i) \bmod n \in S \\ 0, & \text{else.} \end{cases}$$

Provided that $C\langle\{a_1, a_2\}\rangle$ is Hamiltonian, the **two-stripe TSP** is to find a minimum-cost Hamiltonian cycle in the graph $C\langle\{a_1, a_2\}\rangle$. Since $c_{a_1} \leq c_{a_2}$, the problem is to find a Hamiltonian cycle in $C\langle\{a_1, a_2\}\rangle$ using as few edges of length a_2 as possible.

Early work from Burkard and Sandholzer [2] provides necessary and sufficient conditions for $C\langle\{a_1, a_2\}\rangle$ to be Hamiltonian: $GCD(n, a_1, a_2)$ must equal 1. That this condition is necessary follows directly: If $GCD(n, a_1, a_2) := g_2 > 1$, then for any $n, m \in \mathbb{Z}$, $na_1 + ma_2 \equiv_{g_2} 0$. Thus any combination of edges of length a_1 and a_2 will remain within the vertices $\{v \in [n] : v \equiv_{g_2} 0\}$, which is a strict subset of $[n]$. We will constructively show tours demonstrating sufficiency for the two-stripe case. We state the result of Burkard and Sandholzer for general circulant TSP below; we are again primarily interested in the cases where $t \in \{1, 2\}$.

Proposition 2.3 (Burkard and Sandholzer [2])**.** *Let* $\{a_1, ..., a_t\} \subset \left[\lfloor \frac{n}{2} \rfloor\right]$ *and let* $\mathcal{G} = \gcd(n, a_1, ..., a_t)$*. The circulant graph* $C\langle\{a_1, ..., a_t\}\rangle$ *has* $\mathcal{G}$ *components. The ith component, for* $0 \le i \le \mathcal{G} - 1$*, consists of* $n/\mathcal{G}$ *nodes*

$$\{i + \lambda\mathcal{G} \bmod n : 0 \le \lambda \le \frac{n}{\mathcal{G}} - 1\}.$$

$C\langle\{a_1, ..., a_t\}\rangle$ *is Hamiltonian if and only if* $\mathcal{G} = 1$.

Throughout this paper, we let $g_1 := GCD(n, a_1)$ and $g_2 := GCD(n, a_1, a_2)$. By Proposition 2.3, an instance to the two-stripe TSP has a solution with finite cost if and only if $g_2 = 1$.

Circulant graphs have a rich structure that allows us to understand $C\langle\{a_1, a_2\}\rangle$ in terms of $C\langle\{a_1\}\rangle$. First, by Proposition 2.3, the graph $C\langle\{a_1\}\rangle$ consists of g_1 components. Each component is a cycle of size $\frac{n}{g_1}$: Start at some vertex i for $0 \le i < g_1$, and continue following edges of length a_1 until reaching $i + \left(\frac{n}{g_1} - 1\right) a_1$. Taking one more length-a_1 edge wraps back to vertex i, since

$$i + \frac{n}{g_1} a_1 = i + n\frac{a_1}{g_1} \equiv_n i$$

(where the final equivalence follows because g_1 divides a_1 by definition)[2].

Provided $g_2 = 1$, Proposition 2.3 indicates $C\langle\{a_1, a_2\}\rangle$ is Hamiltonian (and therefore connected). Thus edges of length a_2 connect the components of $C\langle\{a_1\}\rangle$. They do so, moreover, in an extremely structured manner. Consider, e.g., Figure 2, which shows $C\langle\{a_1, a_2\}\rangle$ for three different two-stripe instances on $n = 12$ vertices. In all cases, $a_1 = 3$. Since $g_1 = GCD(12, 3) = 3$, there are three components of $C\langle\{a_1\}\rangle$ in all instances (specifically, $\{0, 3, 6, 9\}, \{1, 4, 7, 10\}$, and $\{2, 5, 8, 11\}$). The instances are drawn so that each component is in its own column. While the columns are drawn in slightly different orders for each instance, and while the specific red length-a_2 edges connect different pairs vertices, all three instances are extremely structured. For example, let v be any vertex in the first column. Regardless of the instance or which vertex v is within the first column, $v + a_2$ is always in the second column. Similarly, if v is in the second column, $v + a_2$ is always in the third; if v is in the third column, $v + a_2$ is always in the first.

The following claim from Gutekunst and Williamson [12] makes this structure precise: Consider the graph $C\langle\{a_1, a_2\}\rangle$ and contract each component of $C\langle\{a_1\}\rangle$ into a single vertex. Then, provided $g_2 = 1$, the resulting graph is a cycle. See Lemma 3.6 in Gutekunst and Williamson [12] for a more general statement.

Claim 2.4. *Let* $C_1, C_2, C_3, ..., C_{g_1}$ *denote the connected components of* $C\langle\{a_1\}\rangle$*. Consider the directed graph* $G' = (V', E')$ *on* $V' = [g_1]$ *where* $(u, v) \in E'$ *if and only if there are vertices* $x \in C_u$ *and* $y \in C_v$ *with* $x - y \equiv_n a_2$*. Then* G' *is a directed cycle cover. Moreover, if* $g_2 = 1$*, then* G' *is a directed cycle.*

Proof. First, consider any two vertices x and y in the same component C_i. Then $x \equiv_{g_1} y$. Moreover, $x + a_2 \equiv_{g_1} y + a_2$, so that $x + a_2$ and $y + a_2$ are in the same component Hence, the vertex $i \in V'$ has a single outgoing edge. Analogously $x - a_2 \equiv_{g_1} y - a_2$ so that the vertex $i \in V'$ has a single incoming edge. These facts establish that every vertex of G' has a single outgoing edge and a single incoming edge, so that G' is a directed cycle cover. However, if $g_2 = 1$, G' must also be connected. The only connected, directed cycle cover is a directed cycle. □

Claim 2.4 suggests a convention for drawing two-stripe circulant graphs (used in Figure 2), where each column corresponds to a component of $C\langle\{a_1\}\rangle$. We further arrange the columns (and the ordering of vertices within each column) so that edges of a_2 can be generally drawn horizontally. Specifically, we take the convention that 0 is the top-left vertex. The first column will proceed vertically-down as

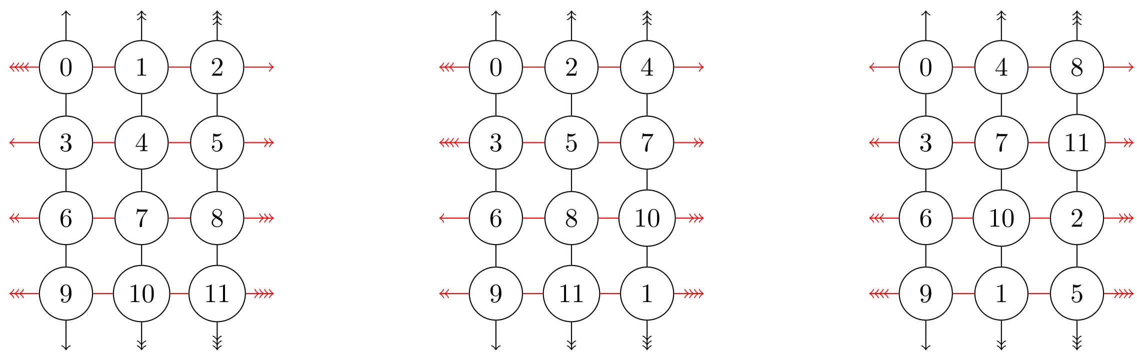


Figure 2: Three drawings of two-stripe instances on $n = 12$ vertices with $a_1 = 3$. In the left instance $a_2 = 1$, in the middle instance $a_2 = 2$, and on the right instance $a_2 = 4$. Black edges are of length a_1 and red edges are of length a_2. Arrows denote horizontal/vertical "wrapping around," and same-colored arrows with the same number of arrowheads wrap to each other. In all instances, e.g., an edge of length a_1 wraps around vertically, connecting 0 (top-left) to 9 (bottom-left). Note that the three cases are almost identical, up to the labelling of the vertices and the way that they wrap around horizontally.

$$\begin{array}{ccccc}
0 & 0 + a_2 & 0 + 2a_2 & \cdots & 0 + (g_1 - 1)a_2 \\
a_1 & a_1 + a_2 & a_1 + 2a_2 & \cdots & a_1 + (g_1 - 1)a_2 \\
2a_1 & 2a_1 + a_2 & 2a_1 + 2a_2 & \cdots & 2a_1 + (g_1 - 1)a_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\left(\frac{n}{g_1} - 1\right) a_1 & \left(\frac{n}{g_1} - 1\right) a_1 + a_2 & \left(\frac{n}{g_1} - 1\right) a_1 + 2a_2 & \cdots & \left(\frac{n}{g_1} - 1\right) a_1 + (g_1 - 1)a_2
\end{array}$$

Figure 3: Our convention for drawing graphs of two-stripe TSP instances. All vertex labels should be implicitly taken modulo n.

$0, a_1, 2a_1, 3a_1, ..., \left(\frac{n}{g_1} - 1\right) a_1$. Our second column "translates" the first column by a_2, so that the top vertex is a_2, the second vertex is $a_1 + a_2$, the third vertex is $2a_1 + a_2$, and so on. Our third column translates by another a_2, and so on. See Figure 3 for our general labeling scheme.

Returning to Figure 2, recall that the three instances have $n = 12$ and $a_1 = 3$. In the left instance $a_2 = 1$, in the middle instance $a_2 = 2$, and in the right instance $a_2 = 4$. Up to a re-labelling of vertices, the only structural change occurs in how the right-most column is connected to the left-most column: the edges that wrap around horizontally, connecting the last column back to the first. On the left, taking an edge of length a_2 from a vertex in the last column wraps around to the first column, but one row lower (2 in the top row connects to 3 in the second row, 5 in the second row connects to 6 in the third row, etc). In the middle, wrapping around shifts down two rows (4 in the top row connects to 6 in the third row, etc). On the right, edges wrap back to the same row.

2.2 Cylinder Graphs

Part of the challenge of two-stripe TSP is the differing ways that horizontal edges wrap around between the first and last column. Our first result, in Section 3, removes this difficulty, and allows us to work on "cylinder graphs": graphs that are similar to those in Figure 2, but without any horizontal edges wrapping around between the last and last column.

[2]In the special case that n is even and $a_1 = n/2$, we think of each of the $g_1 = GCD(n, n/2) = n/2$ components of $C\langle\{a_1\}\rangle$ as a cycle on a pair of vertices.

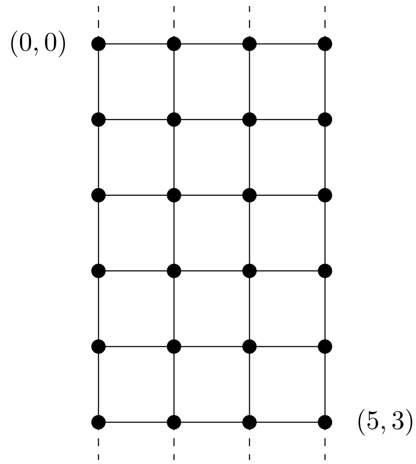


Figure 4: A 6×4 cylinder graph with vertices $(0, 0)$ and $(5, 3)$ indicated. Dashed edges wrap around vertically to create a cylindrical structure.

Definition 2.5. *Let $n = r \times c$. A $r \times c$* **cylinder graph** *is a graph with n vertices arranged into an $r \times c$ grid. For $0 \le i \le r-1$ and $0 \le j \le c-1$, a vertex in row i and column j is adjacent to:*

- *The vertex in row i and column $j+1$, provided $j < c-1$,*
- *The vertex in row i and column $j-1$, provided $j > 0$,*
- *The vertex in row $i-1 \mod r$ and column j, and the vertex in row $i+1 \mod r$ and column j.*

See, e.g., Figure 4. It will often be helpful to refer to a vertex in row x and column y as (x, y), indexed by its column and row, starting from 0. We take the convention that the top-left vertex is $(0, 0)$. Hence, the bottom-right vertex is $(r-1, c-1)$. In general, a vertex $v = xa_1 + ya_2$ will have cylindrical coordinates (x, y). We similarly treat the cylinder graph as inheriting the cost structure of a two-stripe circulant instance. That is, all the vertical edges have cost c_{a_1}, and all the horizontal edges have cost c_{a_2}.

2.3 Conventions for the Two-Stripe TSP

We note several cases where the two-stripe TSP is trivial:

- If $g_2 > 1$, Proposition 2.3 implies that $G\langle\{a_1, a_2\}\rangle$ is not Hamiltonian.
- If $g_1 = 1$, then, by Proposition 2.3, the graph $G\langle\{a_1\}\rangle$ is Hamiltonian. Hence, since $c_{a_1} \le c_{a_2}$, a cheapest Hamiltonian cycle costs nc_{a_1} and consists of a Hamiltonian cycle on $G\langle\{a_1\}\rangle$. Note that any time n is prime, $g_1 = 1$, so that two-stripe (and indeed, general circulant) TSP is trivial any time n is prime.
- If $c_{a_1} = c_{a_2}$ and $g_2 = 1$, then $G\langle\{a_1, a_2\}\rangle$ is Hamiltonian and any Hamiltonian cycle costs $nc_{a_1} = nc_{a_2}$.

These observations are also made in Greco and Gerace [10] and Gerace and Greco [8]. Hence, we restrict ourselves to the case where $g_1 > g_2 = 1$ and $c_{a_1} < c_{a_2}$. We will also take the convention that $c_{a_1} = 0$ and $c_{a_2} = 1$. In this case, the cost of a Hamiltonian cycle is exactly the number of edges of length-a_2 edges it uses. Following Figure 3, we think of these "expensive" length-a_2 edges as "horizontal edges." The "cheap" length-a_1 edges are analogously considered "vertical edges."

2.4 Main Result

The main result of our paper is a characterization of the optimal tour in any two-stripe circulant TSP instance. We state it below. Section 4 is devoted to its proof, and Section 5 shows how this result naturally leads to a polynomial-time algorithm for two-stripe TSP.

Theorem 2.6. *Let $r = \frac{n}{g_1}$ and $c = g_1$. Suppose the cylindrical coordinates of $-a_2$ are $(x, c-1)$. Let m^* be the smallest integer value of m such that $m \geq -\frac{c}{2}$, and $x \in \{2m+c, -(2m+c)\} \mod r$. Then:*

- *If $m^* \leq 0$, the cost of the optimal tour is c.*
- *If $0 < 2m^* < c-2$, the cost of the optimal tour is $c + 2m^*$.*
- *If $2m^* \geq c-2$ or m^* does not exist, the cost of the optimal tour is $2c-2$.*

2.5 Previous Results on Two-Stripe TSP

Despite the restrictive structure of two-stripe TSP, many of the main previous results are inherited from the more-general circulant TSP. From an approximation-algorithms perspective, the state-of-the-art remains a heuristic from Van der Veen, Van Dal, and Sierksma [20]; on any two-stripe instance, this heuristic provides a tour within a factor of two of the optimal solution (see also Gerace and Greco [7], for a more general 2-approximation algorithm). The performance guarantee is based off of the following combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [20].

Proposition 2.7 (Van der Veen, Van Dal, and Sierksma [20], specialized to two-stripe TSP)**.** *Any Hamiltonian cycle on a two-stripe instance costs at least g_1.*

Proof (sketch). The proof largely follows from Proposition 2.3: Proposition 2.3 (when $g_1 > 1$) guarantees that any Hamiltonian cycle must use at least one edge of length a_2. Deleting this edge yields a Hamiltonian path on $C\langle\{a_1, a_2\}\rangle$. Note that this Hamiltonian path must connect the g_1 components of $C\langle\{a_1\}\rangle$. These components are only connected by length-a_2 edges, and so this Hamiltonian path must use at least $g_1 - 1$ edges of length a_2. In total, then, any Hamiltonian cycle must use at least g_1 edges of length a_2 (i.e., cost at least g_1). □

Proposition 2.7 provides a lower bound on the cost of any Hamiltonian cycle. If a tour of cost g_1 exists, we will refer to it as a "lower bound tour." We can exhibit an upper bound on the optimal solution to any two-stripe instance by providing a feasible tour (as usual, provided that $g_2 = 1$). See Van der Veen, Van Dal, and Sierksma [20] and Gerace and Greco [7] for the more general 2-approximation algorithm we are specializing to the two-stripe TSP. See also Section 6.3 of Gutekunst [11].

Proposition 2.8. *There exists a Hamiltonian cycle of cost $2(g_1 - 1)$ on any two-stripe instance.*

Proof (sketch). Figure 5 indicates Hamiltonian cycles that prove Proposition 2.8, which change slightly depending on the parity of g_1. □

Henceforth, we will refer to the tour in Proposition 2.8 as the "upper bound tour". Proposition 2.8 immediately leads to the aforementioned 2-approximation algorithm: it provides a feasible solution costing at most $2(g_1 - 1) < 2g_1$, which is twice the lower bound of Proposition 2.7 (see Van der Veen, Van Dal, and Sierksma [20] and Gerace and Greco [7] for more general 2-approximation algorithms).

More specific results on the two-stripe TSP come from Greco and Gerace [10] and Gerace and Greco [8]. Both papers provide a sufficient, number-theoretic condition for a tour of cost g_1 to exist. In Theorem 2.6, we strengthen their result to a necessary and sufficient condition for a tour of cost g_1 to exist.

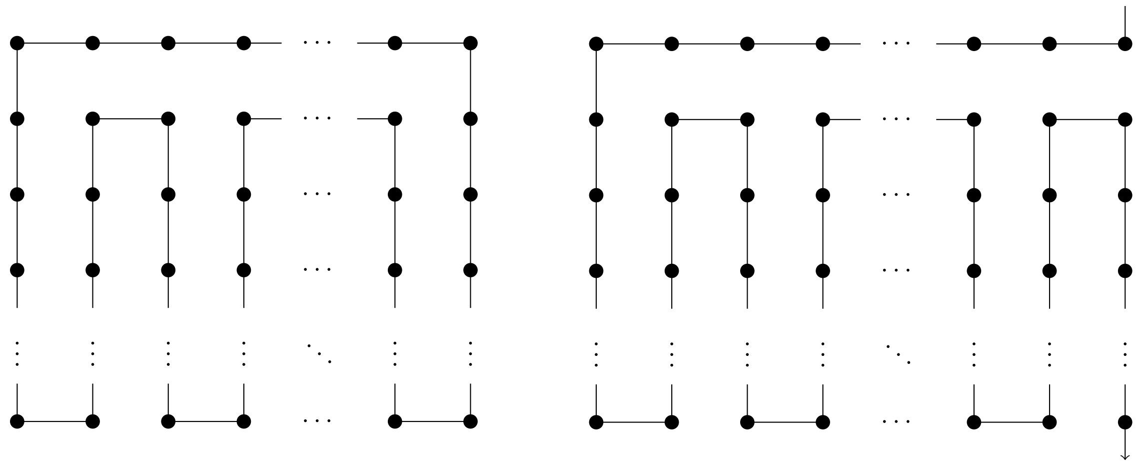


Figure 5: Feasible Hamiltonian cycles of cost $2(g_1 - 1)$ when g_1 is even (left) and odd (right).

Proposition 2.9 (Greco and Gerace [10] and Gerace and Greco [8])**.** *Consider a two-stripe instance, and suppose that there exists an integer y, with $0 \le y \le g_1$, such that*

$$(2y - g_1)a_1 + g_1 a_2 \equiv_n 0.$$

Then the instance admits a Hamiltonian cycle of cost g_1.

In this proof, and throughout this paper, it is often helpful to treat our tours as arbitrarily directed and starting at 0. Doing so allows us to differentiate between an edge of length $+a_i$ (proceeding from v to $v + a_i$) or $-a_i$ (proceeding from v to $v - a_i$). Recall that vertex labels $v \pm a_i$ are implicitly understood to be taken mod n.

Proof (sketch). Suppose such a y exists. We create a Hamiltonian path as follows: Start at 0. In each of the first y columns, we take edges of length $-a_1$ until every vertex in that column is visited (using $\frac{n}{g_1} - 1$ total edges of length $-a_1$ in that column), and then take an edge of length $+a_2$ to move to the next column. In any columns remaining after the first y, take edges of length $+a_1$ until every vertex in that column is visited (using $\frac{n}{g_1} - 1$ total edges of length $+a_1$), and take an edge of length $+a_2$ to move to the next column. We finish this process in the last column, after using $g_1 - 1$ total horizontal edges. See Figure 6.

By construction, this path is Hamiltonian. The final vertex reached is computed as follows: we used $(g_1 - 1)$ edges of length $+a_2$ to get to the final column, we used $\frac{n}{g_1} - 1$ edges of length $-a_1$ in each of y columns, and we used $\frac{n}{g_1} - 1$ edges of length a_1 in the remaining $g_1 - y$ columns. Hence, we end at:

$$\begin{aligned}(g_1 - 1)a_2 - \left(\frac{n}{g_1} - 1\right) y a_1 + (g_1 - y)\left(\frac{n}{g_1} - 1\right) a_1 &= (g_1 - 1)a_2 - 2y\left(\frac{n}{g_1} - 1\right) a_1 + g_1\left(\frac{n}{g_1} - 1\right) a_1 \\ &= (g_1 - 1)a_2 + 2ya_1 - 2yn\frac{a_1}{g_1} + (n - g_1)a_1\end{aligned}$$

Recalling that g_1 divides a_1:

$$\begin{aligned}&\equiv_n (g_1 - 1)a_2 + 2ya_1 - g_1 a_1 \\ &= (g_1 - 1)a_2 + (2y - g_1)a_1.\end{aligned}$$

We now extend our Hamiltonian path from vertex $(g_1 - 1)a_2 + (2y - g_1)a_1$ by taking a horizontal edge. By the assumed conditions on y,

$$(g_1 - 1)a_2 + (2y - g_1)a_1 + a_2 = g_1 a_2 + (2y - g_1)a_1 \equiv_n 0.$$

Hence, this final edge extends our Hamiltonian path to a Hamiltonian cycle, and since it adds one horizontal edge, the resultant cycle costs g_1. $\square$

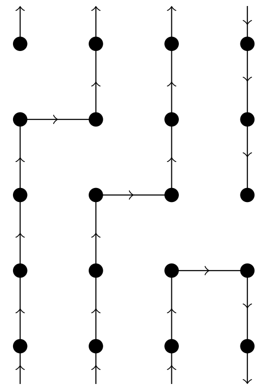


Figure 6: The idea behind Proposition 2.9. The integer y corresponds to the number of columns where arrows point up (corresponding to $-a_1$ edges), so that in this picture, $y = 3$.

Greco and Gerace [10] and Gerace and Greco [8] make two other observations that will be useful to us. These results relate the number of $+a_2$ and $-a_2$ edges used in an optimal Hamiltonian cycle. Let a_+ and a_- respectively denote the number of $+a_2$ and $-a_2$ edges used in an optimal Hamiltonian cycle (again, directed arbitrarily). Note that the cost of the tour is $a_+ + a_-$.

Proposition 2.10 (Greco and Gerace [10] and Gerace and Greco [8])**.** *There exists a minimum-cost Hamiltonan cycle in which* $a_+ - a_- \in \{0, g_1\}$.

Proof (sketch). Consider the graph of $C\langle\{a_1\}\rangle$. Recall that edges of length $\pm a_2$ connect one connected component of this graph to another. Following our convention for drawing graphs of two-stripe instances, each connected components corresponds to a column of the two-stripe graph. Edges of length $+a_2$ join a column to the column on its right (wrapping around from the last column to the first if need be), whereas edges of length $-a_2$ join a column to the column on its left (wrapping around from the first column to the last if need be). Since any Hamiltonian cycle starts and ends in the same column, we have that $a_+ - a_- \equiv_{g_1} 0$. Any Hamiltonian cycle where $|a_+ - a_-| > g_1$ must cost strictly more than $2g_1$ and cannot be optimal by Proposition 2.8. Hence, in a minimum-cost Hamiltonan cycle, $a_+ - a_- \in \{0, g_1, -g_1\}$. If $a_+ - a_- = -g_1$, the tour can be reversed to attain a tour of the same cost, where $a_+ - a_- = g_1$. □

Proposition 2.11 (Greco and Gerace [10] and Gerace and Greco [8])**.** *If* $a_+ = a_-$ *in a minimum-cost Hamiltonian cycle, then the minimum-cost Hamiltonian cycle costs* $2(g_1 - 1)$.

Proof (sketch). We analogize an argument from Claim 4.3 of Gutekunst and Williamson [12]. Consider the columns of our circulant graph (i.e., the components of $C\langle\{a_1\}\rangle$). Each edge of length $+a_2$ moves one column to the right, while edges of length $-a_2$ move us one column to the left (in both cases, wrapping from column $g_1 - 1$ to 0 or vice versa when necessary). A Hamiltonian tour must visit all columns, and the only way we move between columns is using edges of length $+a_2$ and $-a_2$. If we view the Hamiltonian cycle as a sequence of edges $e_1, ..., e_n$, we can delete all the edges of length $\pm a_1$ to attain a subsequence $L' = (e_{i_1}, ..., e_{i_\ell})$ where $i_1 < i_2 < \cdots i_\ell, e_{i_j} \in \{\pm a_2\}$. We upper bound the number of columns visited as follows: Set $U = 1$, corresponding to starting at some column. Until L' is either all $+a_2$s or all $-a_2$s, find an occurrence of a $+a_2$ followed by a $-a_2$ in L' (or a $-a_2$ followed by an $+a_2$); delete these two elements and increment U by 1. Once this process terminates, increment U by $|L'|$ (the number of $+a_2$s or $-a_2$s remaining when L' is either all $+a_2$s or all $-a_2$s). Note that, at the end, $U = \max\{a_+, a_-\} + 1$. U provides an upper bound on the number of columns visited: Any time an $+a_2$ is followed by a $-a_2$ in L' (or vice versa), the effect is to move to one adjacent column and then move back. Hence we visit at most one new column. Thus in order to visit the g_1 columns, we need $U = g_1$, so if $a_+ = a_-$, then we must have $a_+ = a_- = g_1 - 1$. □

Based on the above results, Greco and Gerace [10] and Gerace and Greco [8] are able to identify a few types of instances for which two-stripe TSP is solvable. For example, if $g_1 = 2$, then the upper and lower bounds match, so that an upper bound tour is optimal. Finally, they present their main theorem, which provides a sufficient (but not necessary) condition for the upper bound to be optimal, and constructs an additional tour. We state it without proof below, as we will provide a full resolution to two-stripe TSP that does not rely on it.

Proposition 2.12 (Theorem 4.4 in Gerace and Greco [8])**.** *Let*

$$S = \left\{ y : 0 \le y < \frac{n}{g_1},\ (2y - g_1)a_1 + g_1 a_2 \equiv_n 0 \right\}.$$

If S is empty, the upper bound of $2(g_1 - 1)$ is optimal. Otherwise, let $y_1 = \min\{S\}$ and $y_2 = \max\{S\}$. Let $m = \min\{y_1 - g_1, \frac{n}{g_1} - y_2\}$. If $m \le 0$, the conditions of Proposition 2.9 are satisfied and the lower bound of g_1 is optimal. Otherwise, there exists a tour of cost $g_1 + 2m$.

Note that the congruence defining S is the same as in Proposition 2.9. Hence, if $\frac{n}{g_1} - 1 \le g_1$, this theorem implies that the optimal solution will always either be the upper-bound tour (if S is empty) or the lower-bound tour (if S is nonempty, in which case any element must be at most g_1 and m will be non-positive).

To prove sufficiency of the upper-bound condition, Greco and Gerace do a small amount of algebra (in the style of Proposition 2.9's proof): For the upper bound to be non-optimal, there must exist a tour where $a_+ - a_- = g_1$. Greco and Gerace argue that any such tour implies that S is non-empty.

Ultimately, the main results of Greco and Gerace [10] and Gerace and Greco [8] provide sufficient (but not necessary) condition for the upper- and lower-bounds in Propositions 2.7 and 2.8. Thus they provide a new, algebraic test that can identify certain instances where the upper- or lower-bounds occur, but leave as open fully characterizing either extreme, as well as determining the optimal solution to any instance where the optimal value is between those bounds.

Finally, Greco and Gerace [10] and Gerace and Greco [8] conjecture that, in cases where the lower bound is not achievable, the optimal solution is either the upper bound or their tours of cost $g_1 + 2m$ identified in their main theorem. Our resolution of the two-stripe TSP confirms this conjecture.

3 GG Paths and a Reduction to Hamiltonian Paths on Cylinder Graphs

In this section, we first introduce specific Hamiltonian paths on cylinder graphs, which we will call *GG paths*. We name these paths after Gerace and Greco ([8], [10]), who conjectured that they are sufficient to describe an optimal tour in settings where the upper bound is not optimal. We then present our first main theorem, Theorem 3.1. This theorem will be the first step in solving the two-stripe TSP. Specifically, Theorem 3.1 reduces the two-stripe TSP to finding a minimum-cost Hamiltonian path (between two specified vertices) on an associated cylinder graph. It will then remain to argue that the optimal Hamiltonian path (between those two specified vertices) on that cylinder graph will always be a GG path.

Theorem 3.1 shows that, to solve two-stripe TSP instances, it suffices to study Hamiltonian paths on cylinder graphs that start in the first column and end in the last column. Without loss of generality, we will assume (and draw) such paths as starting at the top-left vertex: the vertex $(0, 0)$. The cost of such a path will be determined by the number of horizontal edges used. Theorem 4.1 will then be used to argue that, among all Hamiltonian such paths (on a cylindrical graph, starting at $(0, 0)$ and ending at a particular vertex in the last column), there is always an extremely structured optimal Hamiltonian path: a *GG path.*

Figure 7: The start of GG paths using 3 horizontal edges between the first and second column. Note that there are two choices: the first edge goes from $(0,0)$ to $(1,0)$ or wraps vertically from $(0,0)$ to $(r-1,0)$.

We will formally define GG paths in Section 3.2. Informally speaking, a GG path is a path where "all of the funky business" happens between the first and second column: For a Hamiltonian path to start in the first column and end in the last column, it must use an odd number of edges between every pair of consecutive columns. In a GG path, an arbitrary odd number of horizontal edges are used between the first and second column, but then a single edge is used between every remaining pair. Moreover, if a GG path uses $2k+1$ horizontal edges between the first and second column, it will first use $r-2k-1$ vertical edges, and will then alternate between vertical and horizontal edges until all vertices in the first column have been visited. See Figure 7.

3.1 Reduction of 2-stripe problem to cylinder graph problem

We first reduce two-stripe TSP instances to Hamiltonian paths on cylinder graphs that start in the first column and end in the last column. By Proposition 2.8, we can always attain a tour of cost $2(g_1-1)$. Theorem 3.1 says that we can find a cheaper tour costing $k < 2(g_1-1)$ if and only if there there is a corresponding Hamiltonian path on a cylinder graph using $k-1$ horizontal edges.

Theorem 3.1. *Consider a 2-stripe instance with n vertices and edges of length a_1 and a_2. There exists a Hamiltonian cycle costing $k < 2(g_1-1)$ if and only if there is a Hamiltonian path on an $r \times c$ cylinder graph with $c = g_1$ and $r = \frac{n}{g_1}$ using $k-1$ horizontal edges, starting at $(0,0)$, and ending at $(x, c-1)$, where x is the unique solution to*

$$xa_1 \equiv_n -g_1 a_2$$

in $\{0, 1, ..., \frac{n}{g_1} - 1\}$.

We will show that the vertex $(x, c-1)$ in the cylinder graph mentioned in the above theorem corresponds to the vertex $-a_2$ in the original circulant graph. To prove the theorem, we will show, if there is a Hamiltonian cycle that is cheaper than $2(g_1-1)$, it can be "unwrapped" to a $0-(x,c-1)$ Hamiltonian path (where x satisfies the theorem conditions). To do so, we will show that, in any Hamiltonian cycle using fewer than $2(g_1-1)$ horizontal edge, there is some pair of columns with a single length a_2 edge between them. We will then shift vertex labels by multiples of a_1 or a_2 so that we that the edge goes between x, in the last column, and 0, in the first.

Proposition 3.2. *Let $H = (v_0, v_1, v_2, ..., v_{n-1}, v_n = v_0)$ be a Hamiltonian cycle in $C\langle\{a_1, a_2\}\rangle$. Let $H' = (u_0, u_1, ..., u_{n-1}, u_n = 0)$ be the Hamiltonian cycle where $u_i = v_i + a_1$. Then for any two columns C_s and $C_{s+1 \mod g_1}$, the number of horizontal edges between C_s and $C_{s+1 \mod g_1}$ is the same in both H and H'. Moreover, if a horizontal edge is in row k of H, then it is in row $k+1 \mod r$ of H'.*

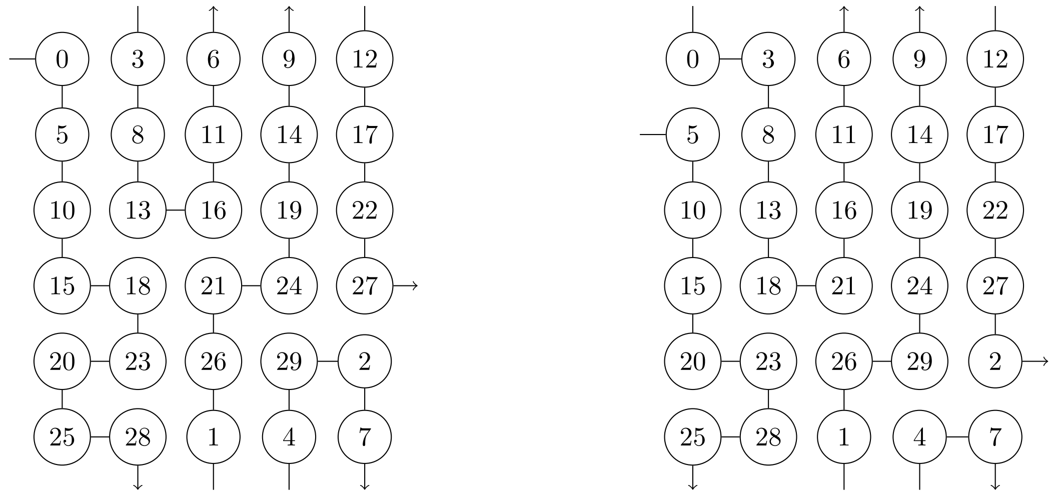


Figure 8: The left image is a Hamiltonian cycle in a two-stripe instance with $n = 30, a_1 = 5$ and $a_2 = 3$. The right shows the effect of adding a_1 to every vertex label (so that, e.g., the edge between 15 and 18 becomes an edge between 20 and 23).

See, e.g, Figure 8, where every vertex in left cycle H has its label shifted by a_1 to get the right cycle (e.g. the edge between 15 and 18 is now between 20 and 23, one row lower). Recall also that the columns of $C\langle\{a_1, a_2\}\rangle$ are the components of $C\langle\{a_1\}\rangle$.

Proof. The first statement follows from the fact that,

$$u_i = v_i + a_1 \equiv_{g_1} v_i,$$

since g_1 divides a_1. Hence u_i and v_i are in the same column. Further, by our drawing convention, u_i is exactly one row below v_i (wrapping around from row $r - 1$ to row 0 if need be). Note also that any edge from u_i to u_{i+1} has the same length as the edge from v_i to v_{i+1}, so that this shift does not change the total number of horizontal/vertical edges. □

Proposition 3.3. *Let $H = (v_0, v_1, v_2, ..., v_{n-1}, v_n = v_0)$ be a Hamiltonian cycle in $C\langle\{a_1, a_2\}\rangle$. Let $H' = (u_0, u_1, ..., u_{n-1}, u_n = 0)$ be the Hamiltonian cycle where $u_i = v_i + a_2$. Then for any two columns C_s and $C_{s+1 \mod g_1}$, the number of horizontal edges between C_s and $C_{s+1 \mod g_1}$ in H equals the number of horizontal edges between $C_{s+1 \mod g_1}$ and $C_{s+2 \mod g_1}$ in H'*

Proof. Again, any edge from u_i to u_{i+1} has the same length as the edge from v_i to v_{i+1}, so that this shift does not change the total number of horizontal/vertical edges. By construction, if v_i is in column C_s, then $u_i = v_i + a_2$ is in column $C_{s+1 \mod g_1}$. □

Proof (of Theorem 3.1). First, by Proposition 2.8, there always exists a tour of cost $2(g_1 - 1)$. Now suppose we have a tour of cost $k < 2(g_1 - 1)$. We treat this tour as starting at 0 and following in some arbitrary direction, so that it proceeds $v_0 = 0, v_1, v_2, ..., v_{n-1}, v_n = 0$ where $v_i - v_{i-1} \in \{a_1, -a_1, a_2, -a_2\}$. Let a_+ and a_- respectively denote the number of a_2 and $-a_2$ edges in this tour. By Proposition 2.10, we can assume that $a_+ - a_- \in \{0, g_1\}$. By Proposition 2.11, for our tour to have cost strictly less than $2(g_1 - 1)$, we must have that $a_+ - a_- = g_1$.

Our tour starts in column 0. Each $+a_2$ edge moves us one column to the right, and each $-a_2$ edge moves us one column to the left (wrapping around from column 0 or $g_1 - 1$ if need be). Since $a_+ - a_- = g_1$, this tour "wraps around" horizontally: our tour starts in the first column, moving through all columns until the last column, and then uses one more horizontal edge to wrap around to the first column.

We note that there is at most one pair of consecutive columns (possibly the last and first column) between which the tour can use zero horizontal edges: if the tour starts in column 0, didn't use any horizontal edges between columns $i, i+1$ and between $j, j+1$ with $i < j$, the tour would never be able to reach vertices in column $i+1$. Hence, there are at least $g_1 - 1$ pairs of consecutive columns where the tour uses at least one edge. In one of these pairs, the tour must use exactly one edge: if not, the tour would cost at least $2(g_1 - 1) > k$.

By Proposition 3.3, we can shift vertex labels by an arbitrary multiple of a_2 so that this pair of columns is the first and last: it wraps from column $g_1 - 1$ to column 0. By Proposition 3.2, we can further assume that this edge wraps from $-a_2$ (in the last column) to 0 (in the first column): we can shift all vertex labels by some multiple of a_1, until its endpoint in column $g_1 - 1$ is the vertex $-a_2$. Note that shifting the labels of all vertices by the same constant does not change the cost of the tour, by circulant symmetry.

We have now argued that if there is a Hamiltonian cycle costing $k < 2(g_1 - 1)$, the must be a Hamiltonian cycle of cost k with exactly one edge between the last and first column, going from vertex $-a_2$ to vertex 0. Deleting this edge yields a Hamiltonian path on a cylinder graph (with $c = g_1$ and $r = \frac{n}{g_1}$), starting in the first column (at vertex $(0, 0)$) and ending in the last. Moreover, this Hamiltonian path uses one fewer horizontal edge, and so costs $k - 1$. To determine the final vertex $-a_2$ of this Hamiltonian path in terms of the cylindrical coordinates, we recall that vertices in the last column and row x have a label of the form $(g_1 - 1)a_2 + xa_1$. Hence,

$$-a_2 \equiv_n (g_1 - 1)a_2 + xa_1.$$

Rearranging,

$$xa_1 \equiv_n -g_1 a_2.$$

By Proposition 2.1, there is a unique solution x to this equivalence in $\{0, 1, ..., \frac{n}{g_1} - 1\}$. Thus, in cylindrical coordinates, the final vertex is as stated: $(x, c - 1)$.

In all, we have shown that any optimal circulant tour of cost $k < 2(g_1 - 1)$ gives rise to a Hamiltonian path on a cylinder graph with $c = g_1$ and $r = \frac{n}{g_1}$ edges using $k - 1$ horizontal edges, starting at $(0, 0)$, and ending at $(x, c - 1)$.

Completing the proof requires showing that a Hamiltonian path on that cylinder graph, between vertex 0 and $-a_2$, of cost $k - 1 < 2(g_1 - 1) - 1$, gives rise to a Hamiltonian cycle on the circulant instance of cost k. This claim is more direct: our arguments above show that, treating the cylindrical graph as a circulant graph, it has a Hamiltonian path from 0 to $-a_2$ of cost $k - 1$. Adding an edge of length a_2 turns this path into a Hamiltonian tour on a circulant graph of cost k, as desired. □

Remark 3.4. *We are only considering non-trivial instances of two-stripe TSP. Hence we assume that* $g_1 > 1$*, which implies that* $c \geq 2$*. Similarly, because* $g_1 = GCD(n, a_1)$ *and* $a_1 \leq \lfloor \frac{n}{2} \rfloor$*,* $g_1 < n$ *so that* $r \geq 2$*.*

3.2 GG Paths

Recall that GG paths from $(0, 0)$ to the last column are highly structured paths where "all the funky business" happens between the first two columns. See, e.g., Figure 7: GG paths start with a sequence of vertical edges, use all the extra pairs of horizontal edges to alternate between the first two columns, and then use one horizontal edge between every remaining pair of consecutive columns. More formally:

Definition 3.5. *A* **GG path** *with* $2k + 1 + (c - 2)$ *horizontal edges in a cylinder graph is a Hamiltonian path that either:*

1. *Starts at* $(0, 0)$*.*
2. *Then uses* $r - 2k - 1$ *consecutive vertical edges, moving along* $(0, 0), (1, 0), ..., (r - 2k - 1, 0)$*.*

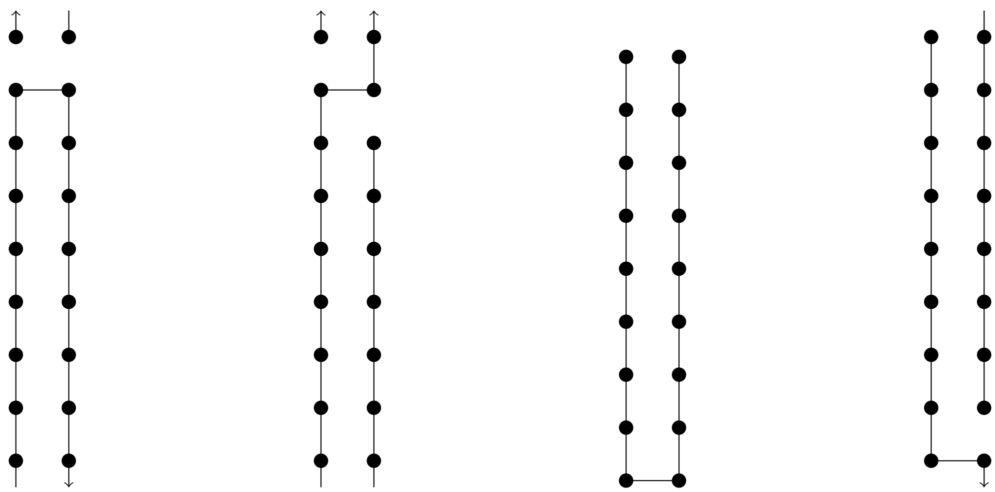


Figure 9: GG paths using 1 horizontal edge between the first and second column.

3. *Then alternates horizontal and vertical edges, using* $2k+1$ *total horizontal edges, moving along* $(r-2k-1,0),(r-2k-1,1),(r-2k,1),(r-2k,0),(r-2k+1,0),...,(r-1,0),(r-1,1)$ *and ending at* $(r-1,1)$.
4. *Then traverses the remaining vertices in the graph, by traversing a column, taking a horizontal edge to the next column, then traversing that column, and so on until all the vertices have been visited. This part of the path ends in the last column and uses* $c-2$ *additional horizontal edges (one between every remaining pair of consecutive columns).*

or

1. *Starts at* $(0,0)$.
2. *Then uses* $r-2k-1$ *consecutive vertical edges, moving along* $(0,0),(r-1,0),(r-2,0),...(2k+1,0)$.
3. *Then alternates horizontal and vertical edges, using* $2k+1$ *total horizontal edges, moving along* $(2k+1,0),(2k+1,1),(2k,1),(2k,0),...,(1,0),(1,1)$ *and ending at* $(1,1)$.
4. *Then traverses the remaining vertices in the graph, by traversing a column, taking a horizontal edge to the next column, then traversing that column, and so on until all the vertices have been visited. This part of the path ends in the last column and uses* $c-2$ *additional horizontal edges (one between every remaining pair of consecutive columns).*

See Figure 7.

In a GG path with $2k+1$ horizontal edges between the first and second column, and where $k \geq 1$, there are exactly two options for how to start the tour: the first edge goes from $(0,0)$ to $(1,0)$ or wraps vertically from $(0,0)$ to $(r-1,0)$. Once that choice is made, the edges in the first and second column are fully determined: After following the edges in Definition 3.5, the prescribed part of the path ends at either $(1,1)$ or $(r-1,1)$. Suppose the sequence ends at $(r-1,1)$, as in the left of Figure 7. Then, since $k \geq 1$, the vertex $(r-2,1)$ must have degree 2 and the only possible edge goes from $(r-1,1)$ to $(0,1)$. If the path ends at $(r-1,1)$, the situation is analogous. When $k=0$, there are four options as shown in Figure 9.

Because GG paths only use one horizontal edge between consecutive columns starting from the second column onward, they are extremely structured. Moreover, when moving to a new column (starting at the

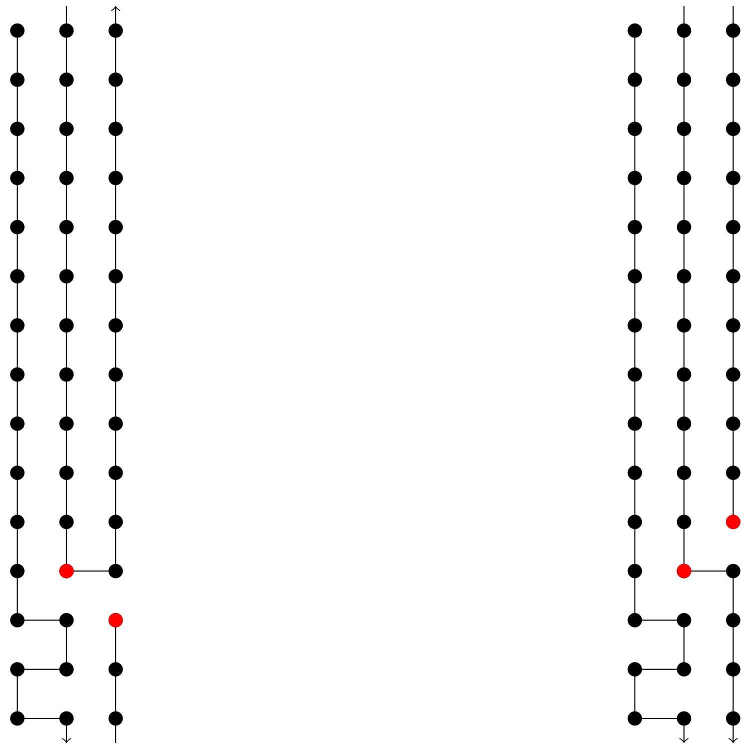


Figure 10: Extending a GG path on 2 columns to a GG path on 3 columns. Notice that, when entering the third column, there are exactly two choices: the first vertical edge in the third column either moves up or down. From there, the third column is completely determined. The last vertex visited in the second and third columns are shown in red. Note that the red vertex in the third column is exactly one row away from the the red vertex in the second column.

third column), a GG path has exactly two options: the first edge in that column vertically is either vertically "up" (going from (i, j) to $(i-1, j)$) or vertically "down" (going from (i, j) to $(i+1, j)$). The net effect is that, if the last vertex visited in the second column is $(i, j-1)$, the last vertex visited in the third column will be $(i+1, j)$ or $(i-1, j)$. See Figure 10.

This structure of GG paths applies inductively. Starting in the third column, each time a GG path enters a new column, it has the exact same two options. Tracing out this process allows us to quickly determine where a GG path can end, based on its path through the first two columns. See Figure 11.

We will argue that, in solving the minimum-cost Hamiltonian path problem on a cylinder graph between $(0, 0)$ and a vertex in the last column, it suffices to consider GG paths: if a minimum-cost path between $(0, 0)$ and $(i, c-1)$ requires k horizontal edges for any $0 \le i \le r-1$, there is a GG path from $(0, 0)$ to $(i, c-1)$ using at most k horizontal edges. Hence, we characterize exactly where a GG path, with some fixed number of horizontal edges, can end up. We define this set below.

Definition 3.6. *Let $A_{r,c,m}$ denote the set of row indices of vertices in the last column reachable on a GG path in an $r \times c$ cylinder graph using* **at most** $2m + (c-1)$ *total horizontal edges.*

In the definition of $A_{r,c,m}$, it is useful to think of the m as the number of extra "expensive pairs" of horizontal edges used between the first and second column. To get $A_{15,c,1}$ for $2 \le c \le 6$, e.g., we would consider the row indices of both the red vertices in Figure 11 and any vertices we get from the tours in Figure 9. Note also that a GG Path can have at most $\lfloor \frac{r-1}{2} \rfloor$ extra horizontal pairs, because the extra horizontal edges are all between the first two columns.

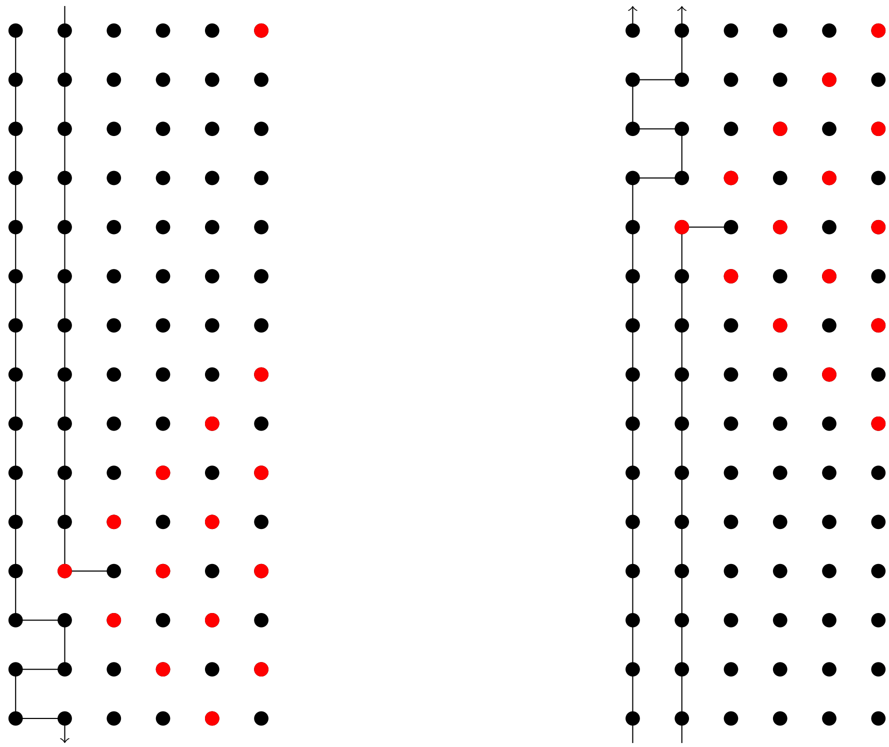


Figure 11: Possible ending vertices for a GG path with 3 edges between the first two columns.

Proposition 3.7. *Let* $m \leq \lfloor \frac{r-1}{2} \rfloor$. *Then*

$$A_{r,c,m} = \{c + 2m - 2i \mod r : 0 \leq i \leq c + 2m\}.$$

Note that, by our definition of $A_{r,c,m}$, this means that the set of vertices reachable in a GG path on an $r \times c$ cylinder graph using at most $2m + (c-1)$ total horizontal edges is:

$$\{(c + 2m - 2i \mod r,\ c - 1) : 0 \leq i \leq c + 2m\}.$$

Our proof follows the structure of GG paths discussed above: we need only identify the vertices reachable when $c = 2$. Then we follow the "triangular growth" of the red vertices in Figure 11, where a GG path ending in some row k of column c corresponds to a GG path ending in column $c+1$ in row $k-1$ or $k+1$ (taken mod r).

Proof. Our proof proceeds using two rounds of induction. By definition, $A_{r,c,m} \subset A_{r,c,m+1}$. We first induct on m to characterize $A_{r,2,m}$. Then we induct on c to be fully general.

Hence, consider $A_{r,2,0}$. These are exactly the paths in Figure 9. We can thus see that

$$A_{r,2,0} = \{0, 2, r-2\}.$$

This matches the proposition statement.

Now we induct on m. For $m \geq 1$, by definition $A_{r,2,m}$ consists of the vertices in $A_{r,2,m-1}$, together with the vertices that can be reached using a GG path with exactly $2m+1$ horizontal edges. Here it is useful to view Figure 7. There are exactly two GG paths with $2m$ extra horizontal edges that we need to consider, corresponding to the first edge going "down" from $(0,0)$ to $(1,0)$, or to the first edge going "up" and wrapping vertically from $(0,0)$ to $(r-1,0)$. In the former, we use all $2m+1$ horizontal edges at rows $r-1, r-2, ..., r-(2m+1)$ and thus end at $(r-2m-2,1)$. In the latter, we use horizontal edges at rows $1, 2, ..., 2m+1$ and end at $(2m+2,1)$. Hence

$$\begin{aligned}
A_{r,2,m} &= A_{r,2,m-1} \cup \{r-2m-2\} \cup \{2m+2\} \\
&= \{2+2(m-1)-2i \mod r : 0 \le i \le 2+2(m-1)\} \cup \{r-2m-2\} \cup \{2m+2\} \\
&= \{2m, 2m-2, \dots, -2m+2, -2m\} \cup \{-(2m+2)\} \cup \{2m+2\} \mod r \\
&= \{2+2m-2i \mod r : 0 \le i \le 2+2m\}.
\end{aligned}$$

Now that we have established the proposition for $A_{r,2,m}$, we induct on c to prove the proposition for $A_{r,c,m}$. For $c \ge 2$, we have that

$$A_{r,c+1,m} = \{x \pm 1 : x \in A_{r,c,m}\}.$$

This is the "triangular growth" of the red vertices in Figure 11, i.e., because we can only use one horizontal edge between columns $c-1$ and c. Thus by induction, since $A_{r,c,m} = \{c+2m-2i \mod r : 0 \le i \le c+2m\}$, we have that

$$A_{r,c+1,m} = \{c+2m-2i \pm 1 \mod r : 0 \le i \le c+2m\}.$$

We now trace through the values of $c+2m-2i \pm 1$ as i ranges from 0 to $c+2m$. Since $c+2m-2i-1 = c+2m-2(i+1)+1$, we see that the values trace through

$$c+2m-2(0)+1, c+2m-2(0)-1, c+2m-2(1)-1, c+2m-2(2)-1, ..., c+2m-2(c+2m)-1.$$

That is,

$$(c+1)+2m-2(0), (c+1)+2m-2, (c+1)+2m-4, ..., (c+1)+2m-2(c+1+2m).$$

That is, $\{(c+1)+2m-2i : 0 \le i \le c+1+2m\}$. This inductively establishes that $A_{r,c+1,m}$ is correctly stated.
□

The above characterization of $A_{r,c,m}$ immediately implies the following corollary, which essentially states that for each additional pair of horizontal edges used, the number of vertices reachable in the last column increases by at most 2. Furthermore, we can characterize exactly which new vertices (if any), become reachable.

Corollary 3.8. *We have* $A_{r,c,m} \setminus A_{r,c,m-1} \subseteq \{c+2m, -(c+2m)\} \mod r$.

Remark 3.9. *Our argument in Theorem 5.1 will be a minimal-counterexample style argument that shows that any minimum-cost Hamiltonian path in a cylinder graph (starting at $(0,0)$ and ending in the last column) can be made iteratively closer a GG path. To do so, we will consider the number of horizontal edges between pairs of adjacent columns. Specifically, we consider these from the rightmost column to the left: the sequence $(h_{g_1-2}, h_{g_1-3}, ..., h_1, h_0)$, where h_i is the number of horizontal edges used in P between the ith and $(i+1)$st columns. Note that, for GG paths, this sequence is of the form $(1, 1, ..., 1, 2m+1)$ and so is lexicographically minimal. As part of our base case, we know that GG paths capture all such paths that only use extra horizontal edges between the first two columns.*

Proposition 3.10. *Let P be a Hamiltonian path in an r by c cylinder graph, starting at $(0,0)$ and ending at $(x, c-1)$ for some row x. Suppose that P uses $2m+1$ horizontal edges between the first pair of columns, and then 1 horizontal edge between every subsequent pair of columns. Then $x \in A_{r,c,m}$.*

Proof. We have moved the proof of this proposition to Appendix A. □

4 Proof of Main Result

In this section, we restate and prove our main result, Theorem 2.6.

Theorem 2.6. *Let $r = \frac{n}{g_1}$ and $c = g_1$. Suppose the cylindrical coordinates of $-a_2$ are $(x, c-1)$. Let m^* be the smallest integer value of m such that $m \geq -\frac{c}{2}$, and $x \in \{2m+c, -(2m+c)\} \mod r$. Then:*

- *If $m^* \leq 0$, the cost of the optimal tour is c.*
- *If $0 < 2m^* < c-2$, the cost of the optimal tour is $c + 2m^*$.*
- *If $2m^* \geq c-2$ or m^* does not exist, the cost of the optimal tour is $2c-2$.*

Our proof of Theorem 2.6 relies on Theorem 4.1. This is one of the main technical results of the paper, and we devote Section 6 to its proof. We state it without proof below.

Theorem 4.1. *Consider a cylinder graph on $n = r \times c$ vertices, with r rows and c columns. Suppose we have a Hamiltonian path, starting at 0 and ending in the last column, and suppose it uses at most $(c-1)+2m$ horizontal edges. Then it must end at a row in $A_{r,c,m}$.*

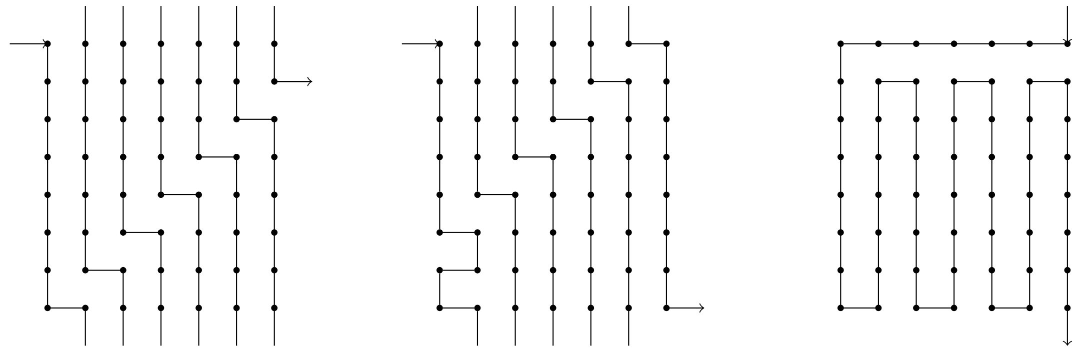
With Theorem 4.1 in hand, we can prove Theorem 2.6.

Proof of Theorem 2.6. Suppose $m^* \leq 0$. This implies $x \in \{-c, -c+2, \ldots, c-2, c\}$. Note that this set is equal to $A_{r,c,0}$ by Proposition 3.7. By definition of $A_{r,c,0}$, there exists a GG path, using exactly $c-1$ horizontal edges, from 0 to $-a_2$. Appending the edge from $-a_2$ to 0 gives a Hamiltonian cycle with cost c. This cycle is optimal, since c is a lower bound on the cost of any tour.

Next, suppose $0 < 2m^* < c-2$. Again, by the characterization of $A_{r,c,m}$, in Proposition 3.7 we have that $x \in A_{r,c,m^*}$. Moreover, by the minimality of m^*, we know that m^* is the *smallest* value of m for which $x \in A_{r,c,m}$. By definition of A_{r,c,m^*}, there is a GG path from 0 to $-a_2$ with cost $(c-1)+2m^*$, and appending the wraparound edge from $-a_2$ to 0 gives a tour of cost $c+2m^*$. To show that this is optimal, note first that since $2m^* < c-2$, we have $c+2m^* < 2(c-1)$. Therefore by Theorem 3.1, the optimal tour consists of a shortest Hamiltonian path from 0 to $-a_2$, plus the wraparound edge from $-a_2$ to 0. Now, by Theorem 4.1, there exists a shortest Hamiltonian path that is a GG path. Since m^* is the smallest value of m for which $x \in A_{r,c,m}$ it follows that the shortest GG path from 0 to $-a_2$ has cost $(c-1)+2m^*$, so we are done.

Finally, suppose that $2m^* \geq c-2$ or m^* does not exist. This implies that $x \notin A_{r,c,m}$ for any m with $0 \leq 2m < c-2$. This, together with Theorems 3.1 and 4.1, implies that the upper bound tour is optimal, the cost of which is $2c-2$. □

Figure 12 shows the three types of tours that correspond to the three cases in Theorem 2.6.



(a) A lower bound tour, which has cost c.

(b) A tour that consists of a GG path using m additional horizontal pairs, plus the wraparound edge to 0. Here, $m = 1$.

(c) An upper bound tour, which has cost $2c - 2$.

Figure 12: An illustration of the three types of tours in Theorem 2.6.

5 The Algorithm

We turn now to showing how Theorem 2.6 naturally leads to an algorithm for two-stripe circulant TSP. In this section, we will describe the algorithm formally and prove that it runs in polynomial time. As mentioned in the introduction, it is possible to represent the input to two-stripe circulant TSP with 3 numbers:

- n, the number of nodes,
- $a_1, a_2 \leq \frac{n}{2}$, the lengths of the edges with finite cost (i.e. the cost of edge ij is finite iff $\min\{n - |i-j|, |i-j|\} \in \{a_1, a_2\}$).

(Recall that we assume without loss of generality that $c_{a_1} = 0$ and $c_{a_1} = 1$.)

It is important to discuss what we mean by "polynomial time". We will take "polynomial time" to mean polynomial time in the bit size of the input, which in our case is *polylogarithmic* in n. Note then that we cannot, strictly speaking, output the entire tour as a sequence of vertices, because this would require listing n numbers, taking $\Omega(n)$ time. Instead, we will show a guarantee on our algorithm that is similar in spirit to the statement of Theorem 2.6: There are two parametrized classes of tours to which the optimal tour can belong, and we can determine the class as well as output a set of parameters that describe the tour in polynomial time. (In particular, we are able to compute the *cost* of the optimal tour in polynomial time.)

Below is the algorithm. It mimics the statement of Theorem 2.6. Throughout the remainder of this section, we will let $r := \frac{n}{g_1}$ and $c := g_1$.

1. Calculate the row index of $-a_2$. That is, calculate the value of $x \in \{0, 1, \ldots, r-1\}$ such that the cylindrical coordinates of $-a_2$ are $(x, c-1)$. (We will explain how to do this step in $O(\log^2 n)$ time.)
2. Compute m^*, the smallest integer value of m such that $m \geq -\frac{c}{2}$ and $x \in \{c + 2m, -(c+2m)\} \mod r$. (We will explain how to do this step in $O(\log^2 n)$ time.)
3. If $2m^* < c - 2$, then the cost of the optimal tour is $c + \max\{0, 2m^*\}$. The optimal tour is achieved by taking a cheapest GG path from 0 to $-a_2$ (which will use exactly $2\max\{m^*, 0\} + (c-1)$ horizontal edges), and appending the wraparound edge from $-a_2$ to 0.
4. Otherwise, if $2m^* \geq c - 2$ or m^* does not exist, the cost of the optimal tour is $2c - 2$. In this case, the upper bound tour (see Proposition 2.8 and Figure 5), is optimal.

Theorem 5.1 (Correctness of the Algorithm)**.** *The algorithm correctly determines a min-cost tour.*

Proof. This directly follows from Theorem 2.6 and its proof. □

Theorem 5.2 (Runtime of the Algorithm)**.** *The algorithm calculates the cost of the optimal tour in $O(\log^2 n)$ time. Moreover, it can output a set of parameters that describe an optimal tour in an additional $O(\log^2 n)$ time, and it can return the sequence of n vertices in the tour in $O(n)$ time.*

Proof. To implement the algorithm, we need to first describe how to find the row index x of $-a_2$. We will show how to do this in $O(\log^2 n)$ time. By Theorem 3.1, we know that $0 \le x < \frac{n}{g_1}$ is the unique solution to $a_1 x \equiv -g_1 a_2 \mod n$. Let us see how to compute x in $O(\log^2 n)$ time. Since $g_1 = \gcd(a_1, n)$, we can divide both sides of the above relation by g_1 to obtain the equivalent relation

$$\left(\frac{a_1}{g_1}\right) x \equiv -a_2 \mod \frac{n}{g_1}.$$

Since $\gcd(\frac{a_1}{g_1}, \frac{n}{g_1}) = 1$, this congruence has a unique solution $x \in \{0, 1, \ldots, \frac{n}{g_1}\}$, and it can be found using the extended Euclidean algorithm in $O(\log^2 n)$ time. (See, for example, Theorem 44 in [19].)

In the second step of the algorithm, we need to compute m^*, the smallest integer value of m such that $m \ge -\frac{c}{2}$ and $x \in \{c+2m, -(c+2m)\} \mod r$. Doing this essentially involves solving two congruences, which we can do with the extended Euclidean algorithm in $O(\log^2 n)$ time. The first congruence is $x \equiv c + 2m \mod r$. Letting $r' := r/\gcd(r, 2)$ (so $r' = r$ if r is odd and $r' = r/2$ if r is even), this congruence is either infeasible or has a unique solution $m \in \{0, 1, \ldots, r'\}$, and the extended Euclidean algorithm finds it (or certifies infeasibility) in $O(\log^2 n)$ time. If the congruence is infeasible, we stop and move on. Otherwise, our candidate value for m^* from solving the first congruence is either (1) m if $m - r' < -\frac{c}{2}$, or (2) $m - r'$ if $m - r' \ge -\frac{c}{2}$. Similarly, the second congruence is $x \equiv -(c + 2m) \mod r$. Applying the extended Euclidean algorithm to this second congruence either certifies infeasiblility or gives us a candidate value for m^*. To find the true value of m^*, it suffices to take the minimum of the two candidate values[3]. If both congruences are infeasible, then m^* does not exist, and we are in the case where the upper bound tour is optimal.

So far, we have shown how to compute x and m^* in $O(\log^2 n)$ time. Note that by Theorem 2.6, this is already enough to calculate the cost of the optimal tour. Now, let us see how to output a set of parameters that uniquely describe the tour in $O(\log^2 n)$ time, and how to output the sequence of n vertices that comprise the tour in $O(n)$ time.

1. First, suppose $2m^* \ge c - 2$ or m^* does not exist. Then, as shown in the proof of Theorem 2.6, the optimal tour is the upper bound tour, which uses $2g_1 - 1$ horizontal edges. This tour is described in Proposition 2.8, and does not need any additional parameters to describe. The exact sequence of vertices in the upper bound tour can be listed in $O(n)$ time.

2. On the other hand, suppose $2m^* < c - 2$. Then, the optimal tour consists of a cheapest GG path P from 0 to $-a_2$ (which will use exactly $2m^* + c - 1$ horizontal edges), plus the wraparound edge from $-a_2$ to 0. It remains for us to find parameters that describe this GG path, and show that the path can be listed in $O(n)$ time.

   First, suppose $m^* \ge 1$. In Section 3, we essentially showed that any GG path can be described with 2 parameters:

   (a) The way it moves in the first column (either starting by going vertically up or vertically down). Note that this also uniquely defines how the path moves in the second column.

[3] This technically does not find the smallest possible value of m^* if m^* is negative. But it correctly determines if m^* is negative or not, and if finds the true value of m^* if it is positive, which is all that is needed.

(b) In each of the remaining $c-2$ columns, whether the first vertical edge of P in the column is going up or down. Note that when considering the endpoint of P, the only thing that matters here is the *number* of columns (from the third column onward) for which the first vertical edge of P in the column is going down. We will let k denote the number of columns from the third column onward for which the first vertical edge of P in the column is going up.

If P starts in the first column by going vertically up, then (as in the proof of Proposition 3.7), the last vertex visited by P in the second column is $(2m^* + 2, 1)$. Then, the row index of the last vertex of P (in the last column), is $2m^* + 2 + k - (c - 2 - k) \mod r$. To see where this comes from, let x_i denote the row index of the last vertex that is visited in column i, where $i \geq 2$. Then, any column $i \geq 3$ in which the first vertical edge goes down, we have $x_i = x_{i-1} - 1$, and if the first vertical edge goes up, we have $x_i = x_{i-1} + 1$. Therefore, if exactly k of the columns from the third column onward have their first vertical edge going up (and hence $(c - 2 - k)$ of them have their first vertical edge going down), we know that the row index of the last vertex visited in the last column is $2m^* + 2 + k - (c - 2 - k)$. To solve for k, we just need to find the value of k that makes the above expression equal to $x \mod r$:

$$2k \equiv x + c - 4 - 2m^* \mod r.$$

We can solve the congruence (or certify that no solution exists), using the extended Euclidean algorithm in $O(\log^2 n)$ time. If a solution k is found, then we are done: We know that an optimal GG path starts in the first column by moving up, and starts by going down in k of the columns from the third column onward. This is enough information to output the sequence of n vertices of the tour in $O(n)$ time.

On the other hand, if the linear congruence is infeasible, it must mean that P starts in the first column by going vertically down. In this case (as in the proof of Proposition 3.7), the last vertex visited by P in the second column is $(r - 2m^* - 2, 1)$. Then, the row index of the last vertex of P is $(r - 2m^* - 2) + k - (c - 2 - k) \mod r$. As before, we can solve for k by solving the linear congruence

$$2k \equiv x + 2m^* + c \mod r,$$

which takes $O(\log^2 n)$ time using the extended Euclidean algorithm. With the value of k known, we know that such GG path starts in the first column by moving down, and in the columns from the third onward, starts in k of them by moving down. Again, this allows us to generate the sequence of vertices in the tour in $O(n)$ time.

Note that (at least) one of the two congruences above must always have a solution, since we know that a GG path of cost $2m^* + (c - 1)$ exists.

Finally, we still need to consider the case $m^* \leq 0$. In this case, the optimal GG path uses 1 edge between every pair of consecutive columns. This case is very similar to the case $m^* \geq 1$, so we will omit the details here. The only difference is that when $m^* \leq 0$, there are 3 options for how the GG path can start (as opposed to 2 options when $m^* \geq 1$). These options are described after Definition 3.5 in Section 3.

□

6 Proof of Theorem 4.1

This section contains the final main proof of the paper, which is the proof of Theorem 4.1. Recall the statement of the theorem:

Theorem 4.1. *Consider a cylinder graph on $n = r \times c$ vertices, with r rows and c columns. Suppose we have a Hamiltonian path, starting at 0 and ending in the last column, and suppose it uses at most $(c-1) + 2m$ horizontal edges. Then it must end at a row in $A_{r,c,m}$.*

Our proof of Theorem 4.1 involves a minimal counterexample-style argument. Specifically, we will consider a counterexample to Theorem 4.1 that is minimal in two senses. Suppose that there exists an $r \times c$ cylinder graph with a Hamiltonian path P from $(0, 0)$ to the last column. Suppose further that P uses $(c - 1) + 2m$ horizontal edges, but does not end at a row in $A_{r,c,m}$. Among all such Hamiltonian cylinder graphs and corresponding Hamiltonian paths, we specifically consider an instantiation where:

1. r and c are minimal with respect to $r + c$. We assume that $r, c \geq 2$.
2. Second, among all counterexamples with r rows and c columns, consider a counterexample path P that is minimal with respect to the **reverse-lexicographic ordering** discussed in Remark 3.9: minimal with respect to the lexicographic ordering $(h_{g_1-2}, h_{g_1-3}, ..., h_1, h_0)$, where h_i is the number of horizontal edges used in P between the ith and $(i + 1)$st columns.

As base cases, we note that if $r = 2$ or $c = 2$, then Theorem 4.1 holds: If $r = 2$, then P cannot use any extra horizontal pairs and so must be a GG path. If $c = 2$, then all horizontal edges must be between the first and second column, and this case is handled by Proposition 3.10. Finally, for any r, c, we note that minimal counterexample P cannot have reverse-lexicographic order $(1, 1, 1, \ldots, 2m-1)$ again by Proposition 3.10.

Before diving into the proof, we first devote Section 6.1 to gathering some properties of $A_{r,c,m}$ that will be useful to us later. Sections 6.2 to Section 6.5 will contain the main proof of Theorem 4.1.

6.1 Preliminaries

First, we note some properties of $A_{r,c,m}$ that will be useful later. These largely summarize structural observations of GG paths. See, e.g., Figure 11.

Proposition 6.1. *Let $r, c \geq 2$. $A_{r,c,m}$ satisfies the following properties.*

1. $A_{r,c,m} \subset A_{r,c,m+1}$
2. $i \in A_{r,c,m} \implies i \pm 2 \mod r \in A_{r,c,m+1}$
3. $i \in A_{r,c,m} \implies i \pm 1 \mod r \in A_{r,c+1,m}$.
4. *Let m' be the the largest value of m such that $c + 2m < r$. Then $A_{r,c,m} = A_{r,c,m'}$ for all $m \geq m'$. (If $c > r$, we take $m' = 0$.)*

Proof. The first property follows by definition: vertices in $A_{r,c,m}$ are defined as using at most m extra horizontal pairs, and hence are also reachable using at most $m + 1$ extra horizontal pairs.

For the second property, it suffices to show that if $i \in A_{r,2,m}$, then $i \pm 2 \mod r \in A_{r,2,m+1}$: For any $j \in A_{r,c,m}$, consider a GG path P reaching $(j, c - 1)$ costing at most $2m + 1$. Let $(i, 1)$ be last vertex visited in the second column by P. If $i \pm 2 \mod r$ are in $A_{r,2,m+1}$, then $j \pm 2 \mod r \in A_{r,c,m+1}$.

So, suppose that $i \in A_{r,2,m}$ and is reachable using a GG path P with $2k + 1$ horizontal edges (where $k \leq m$). If $k = 0$, then $i \in \{0, 2, r-2\}$ (see, e.g., Figure 9); but $A_{r,2,1} = \{0, 2, 4, r-4, r-2\}$ (see, e.g., Figures 7 and 9). If $k > 0$ and $r - (2k + 1) \geq 2$, we can consider two GG paths of the same style as P (i.e. starting vertically up if P starts vertically up; starting vertically down otherwise). Considering one such path that uses $k - 1$ extra horizontal pairs of edges and another such path that uses $k + 1$ extra horizontal pairs of edges, we see that they reach $(i \pm 2 \mod r, 1)$. The only remaining cases are where $r - (2k + 1) \in \{0, 1\}$. Figure 13 handles the case when $r - (2k + 1) = 1$ (appealing to cylindrical symmetry if the first edge is vertically up rather than vertically down), and Figure 14 handles the case where $r - (2k + 1) = 0$. In both cases we see that $i \pm 2 \mod r \in A_{r,2,m+1}$.

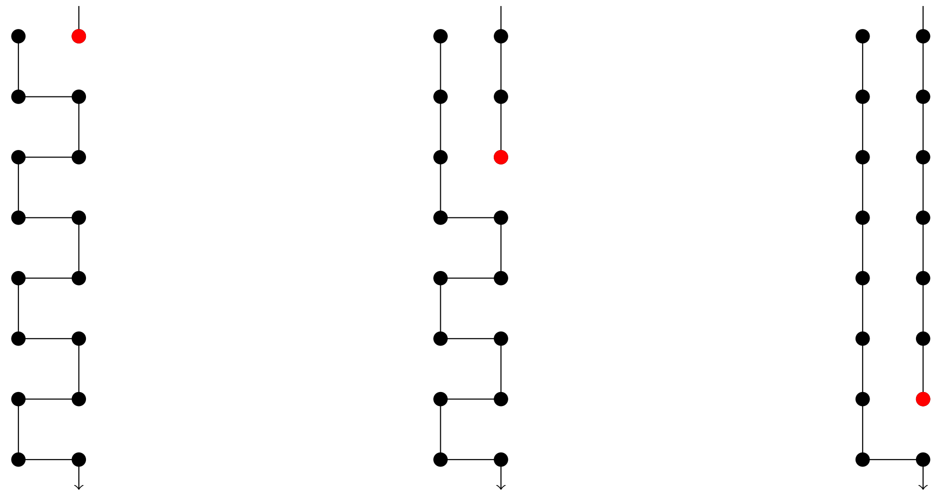


Figure 13: Argument for the third statement of Proposition 6.1 when $r-(2k+1)=1$. If $r-(2k+1)=1$, then GG paths using $k-1$ and 0 extra horizontal pairs respectively reach the vertex two below and two above the end of the GG path.

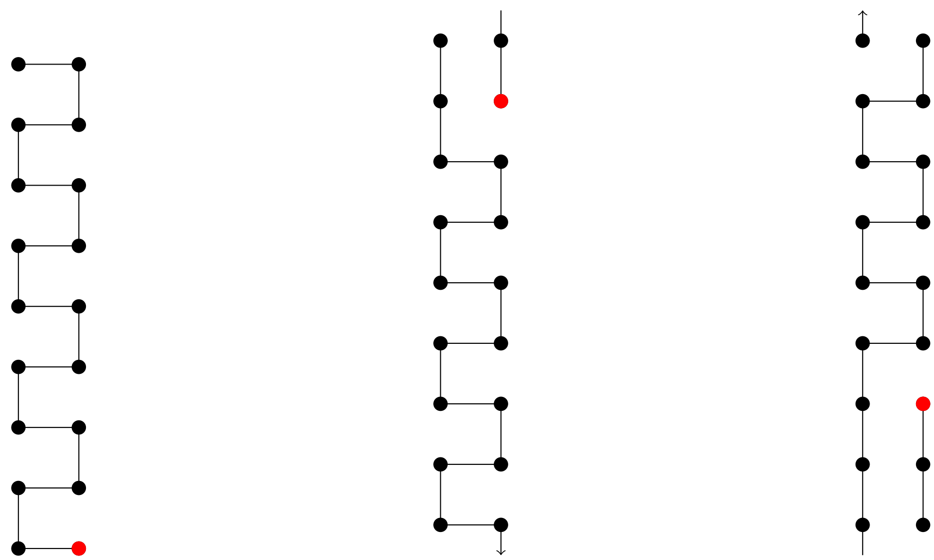


Figure 14: Argument for the third statement of Proposition 6.1 when $r-(2k+1)=0$. If $r-(2k+1)=0$, then GG paths using $k-1$ and $k-2$ extra horizontal pairs respectively reach the vertex two below and two above the end of the GG path. Note that if $k=1$ (and $r=3$), two GG paths using 0 extra horizontal edges should be used instead.

The third property summarizes the structure of GG paths beyond the first and second column: if there is a GG path to $(i,c-1)$, it can be extended to a GG path on $c+1$ columns, ending in row $i\pm 1$: take the same path to $(i,c-1)$, add the edge to (i,c), and then tour through the last column, starting vertically up (and ending at $(i+1 \mod r, c)$) or vertically down (and ending at $(i-1 \mod r, c)$).

Finally, we prove property 4. Let m' be such that $c+2m'<r$ but $c+2(m'+1)\geq r$. Note that

$$A_{r,c,m'}=\{-c-2m',-c-2m'+2,-c-2m'+4,...,c+2m'-2,c+2m'\}.$$

Now consider any $M>m'$. Analogously,

$$A_{r,c,M}=\{-c-2M,-c-2M+2,-c-2M+4,...,c+2M-2,c+2M\}.$$

The only values in $A_{r,c,M}$, but potentially not in $A_{r,c,m'}$, are those of the form $-c-2m'-2i$ and $c+2m'+2i$ for some integer $i>0$. We show that the latter is in $A_{r,c,m'}$; it is analogous to show for the former. To do so, we want to find some $\lambda \in \mathbb{Z}, 0 \le \lambda \le c+2m'$ such that $c+2m'+2i \equiv_r c+2m'-2\lambda$. Doing so shows that $c+2m'+2i$ is already included in $A_{r,c,m'}$.

First suppose that r is even. In this case, we note that, since $c \ge 2$,

$$|(c+2m')-(-c-2m')| = 2c+4m' \ge c+2m'+2 > r.$$

Thus we can subtract some integer multiple of r, say $\mu \times r$, from $c+2m'+2i$ so that

$$c+2m'+2i-\mu r = c+2m'+2i-2\mu\frac{r}{2} \equiv_r c+2m'+2i, \qquad c+2m'+2i-2\mu\frac{r}{2} \in \{-c-2m', c-2m'+2, ..., c+2m'\}.$$

Setting $\lambda = \mu\frac{r}{2}$ yields the desired integer, since r is even. If r is odd, the proof proceeds similarly, but we must subtract a multiple of $2r$. Note that, if r is odd, then $r = c+2m'+1$. We also note that

$$|(c+2m'+1)-(-c-2m'-1)| = 2c+4m'+2 \ge 2r.$$

Hence there must be some integer multiple of $2r$, say $\mu \times 2r$, such that $c+2m'+2i-2\mu r$ is in $[-c-2m'-1, c+2m'+1]$. But $c+2m'+2i-2\mu r \equiv_2 c+2m'$, so we further have that $c+2m'+2i-2\mu r \in [-c-2m', c+2m']$. Once again,

$$c+2m'+2i-2\mu r \equiv_r c+2m'+2i, \qquad c+2m'+2i-2\mu r \in \{-c-2m', c-2m'+2, ..., c+2m'\}.$$

□

Note also that, as a result of our chosen notion of minimality and the observations of Proposition 6.1, we quickly get the following structural result.

Claim 6.2. *In a minimal counterexample to Theorem 4.1, there must be at least 3 edges between the penultimate and final column.*

This claim says that, in the ordering $(h_{g_1-2}, h_{g_1-3}, ..., h_1, h_0)$, the number of edges $h_{g_1-2} \ge 3$.

Proof. Suppose not, and let P be a minimal counterexample. Then P is a Hamiltonian path from $(0,0)$ to the last column on an $r \times c$ cylinder graph, P uses $(c-1)+2m$ horizontal edges, and P ends at a vertex $v \notin A_{r,c,m}$. Moreover, $c \ge 3$. Further, P uses exactly one edge between columns $c-2$ and $c-1$. This edge must be from $(v-1 \mod r, c-2)$ to $(v-1 \mod r, c-1)$ or from $(v+1 \mod r, c-2)$ to $(v+1 \mod r, c-1)$. In either case, deleting that edge and the last column yields a Hamiltonian path P' on an $r \times (c-1)$ cylinder graph, from $(0,0)$ to $(v+1 \mod r, c-2)$ or to $(v-1 \mod r, c-2)$. We assume it is the former, but both cases proceed analogously. By minimality of $r+c$, $v+1 \mod r \in A_{r,c-1,m}$. But by the third statement of Proposition 6.1, this implies that $v \in A_{r,c,m}$, contradicting our assumptions on P. □

6.2 Main Argument: Propagating Cycles to the Left

With Proposition 3.7 and Claim 6.2 in hand, we are ready to begin the main proof of Theorem 4.1. Recall the proof setup described at the beginning of the section: We are assuming a minimal counterexample to Theorem 4.1 with respect to (1) The number of rows plus columns, then (2) the reverse lexicographic order of the number of horizontal edges used between each pair of adjacent columns. Throughout this section, we will abbreviate the edge $\{x,y\}$ as xy.

Let P^* be the Hamiltonian path in the minimal counterexample. Then P^* starts at 0 (the top left vertex), and ends at some vertex $v = (v_1, c-1)$ in the last column. Since P^* is a counterexample to the

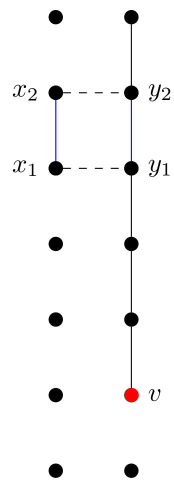


Figure 15: Here, x_1y_1 is the last horizontal edge in P^*, and x_2y_2 is a horizontal edge in P^* immediately above/below it. To begin the cycle propagating argument, we delete x_1y_1 and x_2y_2 (dashed) from P^*. Then, we add the vertical edges x_1x_2 and y_1y_2 (blue). The result uses two fewer horizontal edges, and is either a Hamiltonian path or the union of a path and a cycle.

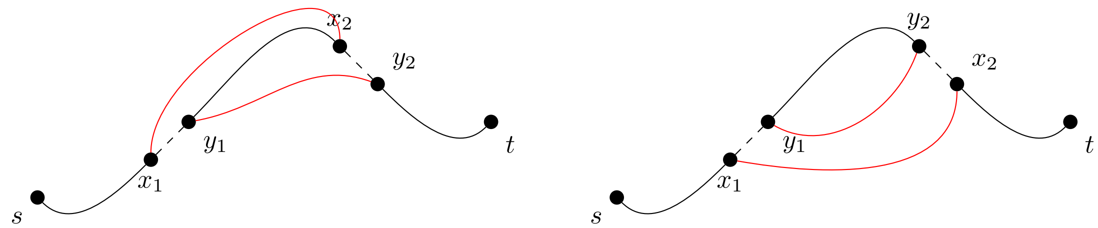
theorem, we have $v_1 \notin A_{r,c,m}$, where m is the number of extra pairs of horizontal edges used by P^*. Our goal is to show that such a minimal counterexample cannot exist by deriving a contradiction. To do this, we use a "cycle propagating" argument, which is described below. Throughout the course of the argument, we will be gradually transforming P^* into structures that more and more resemble GG paths. The intuition behind the argument is to try to push the horizontal edges used by P^* toward the first two columns, all without increasing the cost. If at any point we end up with a Hamiltonian path P that is either (1) cheaper than P^* or (2) has a smaller reverse-lexicographic ordering than P^*, then we will have reached our desired contradiction.

Consider the last horizontal edge in P^*: Call it x_1y_1, where x_1 is the in second-to-last column and y_1 is in the last column. At least one of the two horizontal edges immediately above or below x_1y_1 must also be in P^*. Otherwise, we would have either (1) $v = y_1 - a_2$ and $(y_1, y_1 + a_2) \in P^*$ or $v = y_1 + a_2$ and $(y_1, y_a - a_2) \in P^*$. Since x_1y_1 is the last horizontal edge in P^*, both cases would imply that there is exactly 1 edge between the last two columns, which contradicts Claim 6.2.

So, let x_2y_2 be a horizontal edge that is immediately above or below x_1y_1, where x_2 is in the second-to-last column and y_2 is in the last column. Delete the two adjacent horizontal edges x_1y_1 and x_2y_2, and replace them with the two adjacent vertical edges x_1x_2 and y_1y_2. (See Figure 15.)

The result is that we have used 2 fewer horizontal edges. Moreover, by Claim 6.3 below, we end up with either an 1) 0-v Hamiltonian path, or 2) the disjoint union of a 0-v path and a cycle. Note that if x_1x_2 was an edge in P^* originally, then we have now used x_1x_2 twice; in other words, it is a doubled edge. It is important for the reader to keep in mind that, from now on, the graphs created by our transformation can contain doubled edges. However, because our transformations will preserve the degrees of every node as they were in P^* (so 0 and v will always have degree 1 and every other node will have degree 2), if a doubled edge is ever created, it will be its own component (i.e. a 2-cycle).

In Claim 6.3 below and in what follows, we say a pair of edges p_1p_2, q_1q_2 is "adjacent parallel" if they are either both horizontal and one is immediately above the other, or are both vertical and one is immediately to the right of the other. In other words, p_1p_2 and q_1q_2 are adjacent parallel if either they are both horizontal and $p_1 = q_1 + a_1$, $p_2 = q_2 + a_1$ (or $p_1 = q_1 - a_1$, $p_2 = q_2 - a_1$), or are both vertical and $p_1 = q_1 + a_2$, $p_2 = q_2 + a_2$ (or $p_1 = q_1 - a_2$, $p_2 = q_2 - a_2$).



(a) The case where x_2 is visited before y_2 in P. The black lines (including the two dashed edges) represent P. The solid lines (including the two red edges) represent Q. In this case, Q is an s-t Hamiltonian path.

(b) The case where y_2 is visited before x_2 in P. In this case, Q is the disjoint union of an s-t path and a cycle.

Figure 16: An illustration of the proof of Claim 6.3.

Claim 6.3. *Let P be an s-t Hamiltonian path in a cylinder graph G. Suppose P contains a pair of adjacent parallel edges. Deleting this pair of edges, and replacing by the opposite pair of adjacent edges yields either an s-t Hamiltonian path, or the disjoint union of an s-t path and a cycle.*

Proof. Suppose the pair of edges is horizontal. (The case where they are vertical is similar.) Let the edges be x_1y_1 and x_2y_2, where x_1, x_2 are in the same column and y_1, y_2 are in the same column.

We wish to prove that $Q := P - \{x_1y_1, x_2y_2\} + \{x_1x_2, y_1y_2\}$ is either an s-t Hamiltonian path, or the union of an s-t path and a cycle.

Assume, without loss of generality, that in the s-t path P, x_1 is visited before y_1, which is visited before x_2 or y_2. Referring to Figure 16, it is not hard to see that

- If x_2 is visited before y_2, then Q is an s-t Hamiltonian path, and
- If y_2 is visited before x_2, then Q is the disjoint union of an s-t path and a cycle.

□

Observe that the result cannot be a 0-v Hamiltonian path P. This is because compared to P^*, P uses two fewer horizontal edges and has a smaller reverse-lexicographic ordering. Hence, if P were a 0-v Hamiltonian path, then this would contradict the minimality of P^*. So, the result must be a disjoint union of a 0-v path P and a cycle C.

If a horizontal edge of C is adjacent to a horizontal edge of P, deleting that pair and replacing with the corresponding pair of vertical edges yields a 0-v Hamiltonian path (see Claim 6.4). This Hamiltonian path uses two fewer horizontal edges and has a smaller reverse-lexicographic ordering than P^*, which contradicts the minimality of P^*. So from now on, we may assume without loss of generality that C and P share no adjacent horizontal edges.

Claim 6.4. *Let P be an s-t path and C be a cycle in some G, such that P and C are vertex-disjoint and every vertex of G is contained in either P or C. Let p_1p_2 be an edge of P and let q_1q_2 be an edge of C. Assuming that the edges p_1q_1 and p_2q_2 exist in G, deleting the edges p_1p_2, q_1q_2 from $P \cup C$ and replacing with the edges p_1q_1 and p_2q_2 yields an s-t Hamiltonian path in G.*

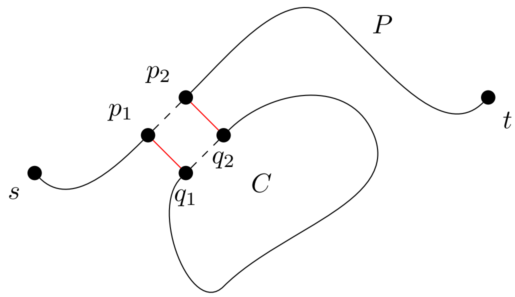


Figure 17: Illustration of the proof of Claim 6.4.

Proof. Without loss of generality, suppose that p_1 is visited before p_2 in P. Then deleting the edges p_1p_2 and q_1q_2 from $P \cup C$ yields the disjoint union of three paths: 1) An s-p_1 path P_1, 2) an p_2-t path P_2, and 3) a q_1-q_2 path Q. Adding the edge p_1q_1 connects P_1 and Q into an s-q_2 path, and adding the edge p_2q_2 connects this path and P_2 into a s-t Hamiltonian path. For an illustration, see Figure 17. □

Consider the leftmost vertical edge in C: Call it v_1v_2. Note that C has at least one vertical edge, because x_1x_2 is one. If there are multiple leftmost vertical edges, break ties by choosing the "topmost" one. (That is, the one that contains the vertex with the smallest row index. Our convention will be that the wraparound edge from the first vertex of a column to the last vertex is the most top edge.) Henceforth, we will refer to this step as selecting the leftmost, topmost vertical edge of C.

Referring to Figure 18a, let $u_1 = v_1 - a_2$, $t_1 = v_1 - 2a_2$, and $w_1 = u_1 - a_1$. Also, let $u_2 = v_2 - a_2$, $t_2 = v_2 - 2a_2$, and $w_2 = u_2 + a_1$. Note that u_1v_1 cannot be an edge in $P \cup C$: If it was, it must be an edge of C since $v_1v_2 \in C$. However, since u_1 is to the left of v_1 and v_2, this would imply that C has some vertical edge that is further to the left compared to v_1v_2, which contradicts our choice of v_1v_2 as a leftmost edge of C.

Furthermore, u_1u_2 cannot be an edge in $P \cup C$. If it were, then it must be in P since it is a vertical edge that is to the left of v_1v_2. In that case, by Claim 6.4, deleting the two vertical edges u_1u_2, v_1v_2 and replacing them with the two horizontal edges u_1v_1, u_2v_2 results in a 0-v Hamiltonian path. Moreover, this Hamiltonian path has a smaller reverse-lexicographic order than P^*, which contradicts the minimality of P^*.

So far, we have established that the edges u_1v_1 and u_1u_2 cannot be present in $P \cup C$. Consider the vertex u_1: It cannot be 0 or v, so it must have degree 2 in $P \cup C$. (It cannot be v because it is not in the last column. It cannot be 0 because if it were, then u_2u_1 and u_2w_2 must be edges in P because (1) u_2 has degree 2 and (2) v_1v_2 is a leftmost edge in C. Now, consider the two adjacent vertical edges u_1u_2 (in P) and v_1v_2 (in C). Deleting this pair and replacing with the corresponding pair of horizontal edges would yield, by Claim 6.4, a 0-v Hamiltonian path. This path has a smaller reverse-lexicographic ordering than P^*, which is a contradiction.) Since u_1 has degree 2 in $P \cup C$, this implies that the two edges u_1w_1 and u_1t_1 are both in $P \cup C$. Moreover, we must have $u_1w_1, u_1t_1 \in P$, because v_1v_2 is the leftmost edge of C.

The same argument shows that $u_2w_2, u_2t_2 \in P$.

Now, perform the following operation: Delete the three edges v_1v_2, t_1u_1, t_2u_2, and replace them with the three edges t_1t_2, u_1v_1, u_2v_2. (Refer to Figure 18b.) We will refer to this operation as the "cycle propagating operation".

Observe that the resulting subgraph uses the same number of horizontal edges as before, and has a strictly smaller reverse-lexicographic order (since the operation pushed two horizontal edges one column to the left). Furthermore, by Claim 6.5, the resulting structure is either 1) a 0-v Hamiltonian path, or 2) the union of an 0-v path with a cycle.

(a) Before. v_1v_2 is the leftmost, topmost edge of the cycle C. The other edges in the figure are part of the path P.

(b) After. The three edges $\{v_1v_2, t_1u_1, t_2u_2\}$ have been deleted and replaced with $\{t_1t_2, u_1v_1, u_2v_2\}$. The result is either a 0-v Hamiltonian path or the disjoint union of a 0-v path and a cycle.

Figure 18: An illustration of one step of the cycle propagating procedure.

Claim 6.5. *After one iteration of the cycle propagating procedure, the resulting subgraph is either 1) a 0-v Hamiltonian path, or 2) the disjoint union of a 0-v path with a cycle.*

Proof. Let Q be the resulting subgraph. Note that the cycle propagating operation does not change the degree of any vertex. Hence, in Q, the degrees of 0 and v are 1, and the degree of every other vertex is 2. This implies that Q is the disjoint union of a 0-v path and zero or more cycles. To prove the claim, we will show that Q consists of at most 2 connected components.

Let $Q' = P \cup C \cup \{u_1v_1, u_2v_2, t_1t_2\}$, so that $Q = Q' - \{t_1u_1, t_2u_2, v_1v_2\}$. (Refer to Figure 18 for an illustration.) Observe that Q' consists of one connected component, because the edges u_1v_1 and u_2v_2 connect P and C. Now, imagine deleting the edges $\{t_1u_1, v_1v_2, t_2u_2\}$ one by one from Q' to obtain Q. t_1u_1 is not a bridge in Q', because it is contained in the cycle $t_1u_1v_1v_2u_2t_2t_1$. Hence $Q' - \{t_1u_1\}$ is connected.

Next, v_1v_2 is not a bridge in $Q' - \{t_1u_1\}$, because it is contained in the cycle C. Hence $Q' - \{t_1u_1, v_1v_2\}$ is connected. Finally, t_2u_2 may be a bridge edge in $Q' - \{t_1u_1, v_1v_2\}$, but deleting it can create at most one new component. Thus Q consists of at most 2 components, which completes the proof.
□

If the result is a 0-v Hamiltonian path P, then we are done. This is because P uses two fewer horizontal edges than P^*, and has a smaller reverse-lexicographic ordering.

Thus, after one step of the cycle propagating procedure, we may assume that the result is the disjoint union of a 0-v path and a cycle. Moreover, observe that the leftmost vertical edge of the new cycle is further to the left than that of the old cycle. (Referring to Figure 18, at least one of the vertical edges t_1t_2, u_1w_1 or u_2w_2 are in the new cycle. All three of these edges are further to the left than v_1v_2, which was a leftmost vertical edge in the old cycle.) Hence, iterating this process will either result in a 0-v Hamiltonian path at some step (in which case we are done), or we get a sequence of paths and cycles $(P_1, C_1), \dots, (P_k, C_k)$ that satisfy the following properties.

Proposition 6.6. *The sequence of paths and cycles $(P_1, C_1), \dots, (P_k, C_k)$ resulting from the cycle propagating procedure satisfy the following properties.*

P1. For each i, P_i is a 0-v path and C_i is a cycle,

P2. For each i, P_i, C_i are vertex-disjoint and together cover all the vertices of the graph,

P3. For each i, P_i and C_i have no horizontal adjacent edges. (Otherwise by Claim 6.4, we could delete them and replace them with the corresponding pair of vertical edges to get a Hamiltonian path P that (1) uses 4 fewer horizontal edges than P^* and (2) has a smaller reverse lexicographic order.)

P4. (P_{i+1}, C_{i+1}) *is obtained from* (P_i, C_i) *by applying the cycle propagating procedure on the leftmost, topmost edge of* C_i*, and*

P5. The last cycle, C_k*, has a vertical edge in the first column.*

Moreover, we have the following property:

P6. For each i*, let* b_i *be the index of the column that contains the leftmost vertical edge of* C_i*. Suppose* $e = \{(a, b_i), (a+1, b_i)\} \in C_i$ *is some vertical edge in* C_i *that is in column* b_i*. Let* $f = \{(a, b_i-1), (a+1, b_i-1)\}$ *be the vertical edge that is immediately to the left of* e*. Then* f *cannot be in* P_i*.*

Proof. Properties P1 – P5 are either self-explanatory or have been proved earlier.

Property P6 is true, because otherwise, we could delete e and f, and replace them with the corresponding pair of horizontal edges. This would give a Hamiltonian path that 1) uses the same number of horizontal edges as P^*, and 2) has a smaller reverse lexicographic order than P^*. This contradicts our choice of minimal counterexample. Note that 2) is true because b_i was the leftmost column containing a vertical edge of C_i, so in particular $b_i \leq c - 2$. □

Note that any or all of the cycles C_i can be 2-cycles (i.e. a doubled edge.) Referring to Figure 18, the only way a doubled edge can be created by the cycle propagating procedure is if before some iteration (see Figure 18a), the edge t_1t_2 is already present in $P \cup C$. Then, after the iteration (see Figure 18b), the edge t_1t_2 is doubled. In particular, this means that only vertical edges can be doubled; a doubled horizontal edge is not possible.

We now look at the sequence of cycles $C_1, C_2, \ldots, C_k$, and consider cases depending on which of them are 2-cycles. In Section 6.3, we will show that C_k must be a 2-cycle. In Section 6.4, we will extend the argument to show that, in fact, every cycle must be a 2-cycle. Finally, we use this structure in Section 6.5 to arrive at a contradiction.

6.3 Last cycle is a 2-cycle

Claim 6.7. *The last cycle* C_k *must be a 2-cycle.*

Proof. Suppose for a contradiction that C_k is not a 2-cycle. Let S be the maximal contiguous set of vertices in the first column such that

1. $0 \in S$, and
2. No vertex in S is in C_k.

Let x and y be the endpoints of S. Let x' and y' be the vertices in the first column, not in S, that are adjacent to x and y respectively. (See Figure 19.) Note that x', y' exist, are distinct, and belong to C_k.

Since C_k is a cycle, both x' and y' have degree 2. Now x' cannot be incident to x in C_k, so it must be incident to $x' + a_2$ and $x' + a_1$ in C_k. (Here, we use the assumption that C_k is not a 2-cycle.)

Consider x. If x has degree 2 in P_k, then it would be incident to $x + a_2$. Then C_k and P_k would have a pair of adjacent horizontal edges, which is leads to a contradiction via Claim 6.4. Therefore x has degree 1 in P_k. The same argument shows that y must have degree 1 in P_k. This is a contradiction, since P_k has exactly one vertex of degree 1 in the first column, namely vertex 0.
□

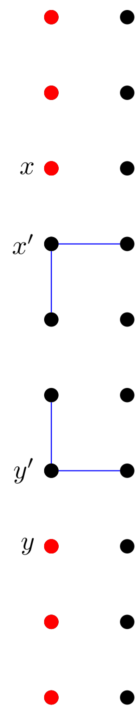


Figure 19: An illustration of the proof of Claim 6.7. The depicted nodes are the first two columns of the graph. The red nodes are the nodes in S. The blue edges are in C_k. Since the path and the cycle cannot have any adjacent horizontal edges, this implies that x and y both have degree 1 in the path, which is impossible.

6.4 Every cycle is a 2-cycle

In the previous subsection, we showed that the last cycle C_k must be a 2-cycle. Next, we will show that in fact every cycle must be a 2-cycle. Throughout this section, when we write the coordinates of a point in a cylinder graph, we will use (i, j) to implicitly mean $(i \mod r, j \mod c)$. This is to make the notation less cluttered.

Claim 6.8. *For all $1 \leq i \leq k$, C_i is a 2-cycle.*

Proof. Suppose for a contradiction that some cycle is not a 2-cycle. We consider the last cycle in the sequence that is not a 2-cycle. That is, let i be the index such that

1. C_i is not a 2-cycle, and
2. $C_{i+1}, \ldots, C_k$ are all 2-cycles.

Note that by the previous claim, $i \leq k-1$. We will show this case cannot happen using an argument similar to the one used in the proof of the previous claim, but is more involved. Essentially, we will argue that many connections in the graph of (P_i, C_i) are forced; that is, there are certain edges we can be sure are present in P_i or C_i. We will then show that the presence of these forced edges implies that two vertices in the first column have degree one in P_i if C_i is not a 2-cycle, which will give our desired contradiction.

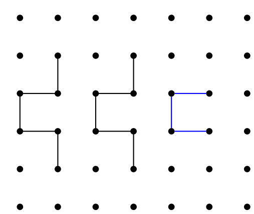
Let the two vertices of C_k be $(a, 0)$ and $(a+1, 0)$, where $0 \leq a \leq r-1$. Note that in fact $1 \leq a \leq r-2$, because $(0, 0)$ cannot be a vertex in C_k; $(0, 0)$ is in P_k. Because $C_k, C_{k-1}, \ldots, C_{i+1}$ are all 2-cycles, it follows from the way we propagated cycles in Section 6.2 that for all $0 \leq j \leq k-i-1$, the following vertices are consecutive on the path P_i (see Figure 20a for an illustration):

$$(a-1, 2j+1),\ (a, 2j+1),\ (a, 2j),\ (a+1, 2j),\ (a+1, 2j+1),\ (a+2, 2j+1)$$

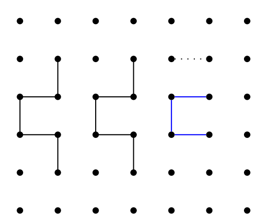
Also, $2(k-i)$ is the index of the leftmost column of C_i, and the edge between the vertices $(a, 2(k-i))$ and $(a+1, 2(k-i))$ is the leftmost, topmost vertical edge of C_i. (Again, see Figure 20a.)

We proceed to argue that certain edges in the graph are forced to be in either P_i or C_i. For convenience, let $b = 2(k-i)$ be the index of the column containing the leftmost edge of C_i. Then the leftmost, topmost vertical edge of C_i is between (a, b) and $(a+1, b)$. (This is the blue edge in Figure 20a.) First, we will show that $a = 1$.

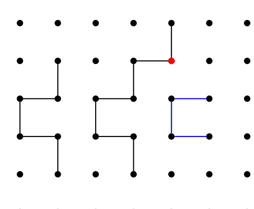
To begin, observe that (a, b) cannot be connected to $(a-1, b)$ in C_i; otherwise, if $\{(a, b), (a-1, b)\}$ were an edge in C_i, then it would be adjacent to the vertical edge $\{(a, b-1), (a-1, b-1)\}$ in P_i. This is impossible, since $\{(a, b-1), (a-1, b-1)\}$ is further to the left than the leftmost edge of C_i, contradicting property P6 in Proposition 6.6. Hence, in C_i, if C_i is not a 2-cycle, then (a, b) must have a horizontal edge to $(a, b+1)$. A similar argument shows that $(a+1, b)$ must have a horizontal edge to $(a+1, b+1)$ in C_i.



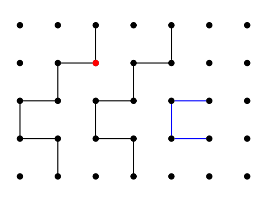
Next, we claim that the edge between $(a-1, b)$ and $(a-1, b+1)$ (the dotted edge in the figure on the right) cannot exist in either P_i or C_i. Indeed, if that edge existed in P_i, then P_i and C_i would share an adjacent pair of horizontal edges, a contradiction. On the other hand, if that edge existed in C_i, then so must the edge between $(a-1, b)$ and $(a-2, b)$, which (since $a \geq 1$), contradicts the fact that $\{(a, b), (a+1, b)\}$ is the leftmost topmost vertical edge in C_i.



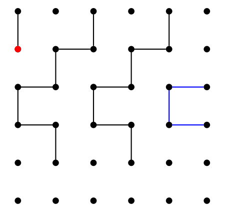
Now, note that $(a-1, b)$ (the red vertex in the diagram to the right) must have degree 2 in $P_i \cup C_i$. Since $(a-1, b)$ cannot have an edge to the vertex to its right (by the previous paragraph), or to the vertex below it (since that vertex already has degree 2), it must have edges to the vertices above it and to its left. In other words, $(a-1, b)$ must have edges to $(a-2, b)$ and $(a-1, b-1)$. Note that the edges $\{(a-1, b), (a-2, b)\}$ and $\{(a-1, b), (a-1, b-1)\}$ must both belong to P_i, because the vertex $(a-1, b-1)$ is in P_i.



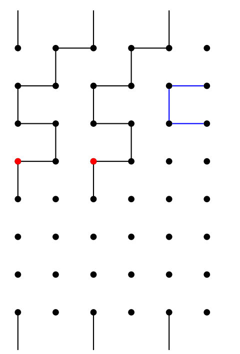
Now, consider the vertex $(a-1, b-2)$ (the red vertex in the diagram to the right). If $b > 2$, then $(a-1, b-2)$ has degree 2 in $P_i \cup C_i$. Since its right neighbor $(a-1, b-1)$ and bottom neighbor $(a, b-2)$ already have degree 2, it must have edges to its top neighbor and left neighbor. As in the previous paragraph, these two edges $\{(a-1, b-2), (a-2, b-2)\}$ and $\{(a-1, b-2), (a-1, b-3)\}$ must both belong to P_i, because the vertex $(a-1, b-3)$ is in P_i. Applying this argument repeatedly, we get that for all $j \in \{b-1, b-3, \ldots, 1\}$, the vertex $(a-1, j)$ has edges to $(a-2, j)$ and $(a-1, j-1)$, and that these edges are all in P_i.



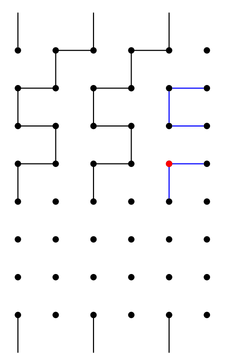
We are now ready to prove that $a = 1$. Consider the vertex $(a-1, 0)$ (red in the diagram to the right). Since it is in the first column, it cannot have any edge to its left. Moreover, its right neighbor $(a-1, 1)$ already has degree 2 by the arguments in the previous paragraph, and its bottom neighbor $(a, 0)$ also already has degree 2 (we knew this at the start of this proof). Therefore, the only possible edge out of $(a-1, 0)$ is the edge to it top neighbor. This implies that $(a-1, 0)$ has degree 1 in $P_i \cup C_i$. This implies $a = 1$, because $(0, 0)$ is the only vertex in the first column with degree 1. Previously we drew our diagrams with $a > 1$ in order to be general, but now that we have shown $a = 1$, we will draw our diagrams with $a = 1$ from now on.



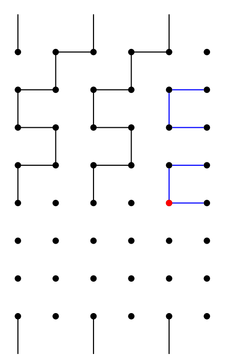
Now, consider the vertex $(3, 0)$. If $r > 3$, then it must have degree 2 in $P_i \cup C_i$. Since it cannot have an edge to $(2, 0)$, it must have edges to $(3, 1)$ and $(4, 0)$. Next, consider the vertex $(3, 2)$. Again, it must have degree 2 in $P_i \cup C_i$. It cannot have edges to $(3, 1)$ or $(2, 2)$, because these already have degree 2. Hence, it must have edges to $(4, 1)$ and $(3, 3)$. Iteratively applying this argument, we get that for all $j \in \{0, 2, \ldots, b-2\}$, the vertex $(3, j)$ (red in the diagram) has edges to $(3, j+1)$ and $(4, j)$.



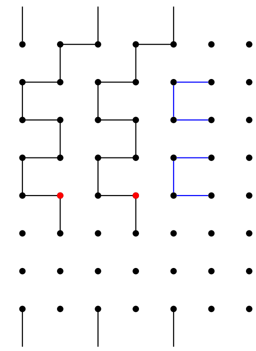
Now, consider the vertex $(3, b)$ (red). It must have degree 2 in $P_i \cup C_i$. However, it cannot have an edge to its top neighbor $(2, b)$ or to its left neighbor $(3, b-1)$, because these already have degree 2. Thus, it must have edges to $(3, b+1)$ and $(4, b)$. If these two edges belonged to P_i, then the horizontal edge $\{(3, b), (3, b+1)\}$ in P_i would be adjacent to the horizontal edge $\{(2, b), (2, b+1)\}$ in C_i, which contradicts property P3 in Proposition 6.6. Therefore, these two edges must belong to C_i.



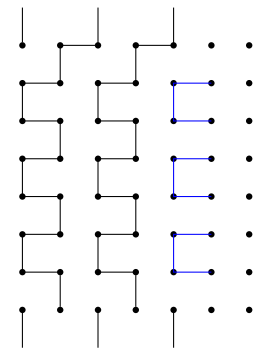
Consider the the vertex $(4, b)$ (red). It is in C_i and must have degree 2. It cannot have an edge going to the left to $(4, b-1)$, since otherwise C_i would have an edge further to the left than its leftmost edge. Also, it cannot have an edge going down to $(5, b)$: If $\{(4, b), (5, b)\}$ were an edge in C_i, then the the edge $\{(4, b-1), (5, b-1)\}$, immediately left to it, would be forced to be in P_i. This would mean that a vertical edge in the leftmost column of C_i is adjacent to a vertical edge of P_i, which contradicts property P6 in Proposition 6.6. Therefore $(4, b)$ must have an edge going to the right to $(4, b+1)$, and this edge is in C_i because $(4, b)$ is in C_i.



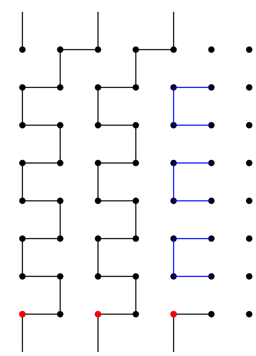
Consider the the vertex $(4, b-1)$. It must have degree 2, and its top and right neighbors are already have degree 2, which implies it has edges to its left neighbor $(4, b-2)$ and its bottom neighbor $(5, b-1)$. These edges are in P_i, because $(4, b-2)$ is. Repeating this argument, we get that for all $j \in \{b-1, b-3, \ldots, 1\}$, the vertex $(4, j)$ (red in the diagram), has edges to $(4, j-1)$ and $(5, j)$, and these edges are all in P_i.



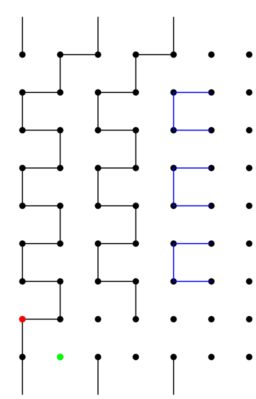
Repeating the argument in the previous 4 paragraphs, we can propagate this structure all the way down the rows. The exact way this structure ends depends on the parity of r. First, suppose r is even. Then the structure is depicted in the diagram on the right. To be precise, we have that for all $l \in \{1, 3, 5, \ldots, r-3\}$, the vertices $(l, b+1), (l, b), (l+1, b), (l+1, b+1)$ are consecutive in C_i. (These are the blue paths of length 3 in the diagram.) Also, for all $l \in \{1, 3, 5, \ldots, r-3\}$ and $j \in \{1, 3, \ldots, b-1\}$, the vertices $(l-1, j), (l, j), (l, j-1), (l+1, j-1), (l+1, j), (l+2, j)$ are consecutive in P_i. (These form the zig-zagging subpaths of P_i going downward.)

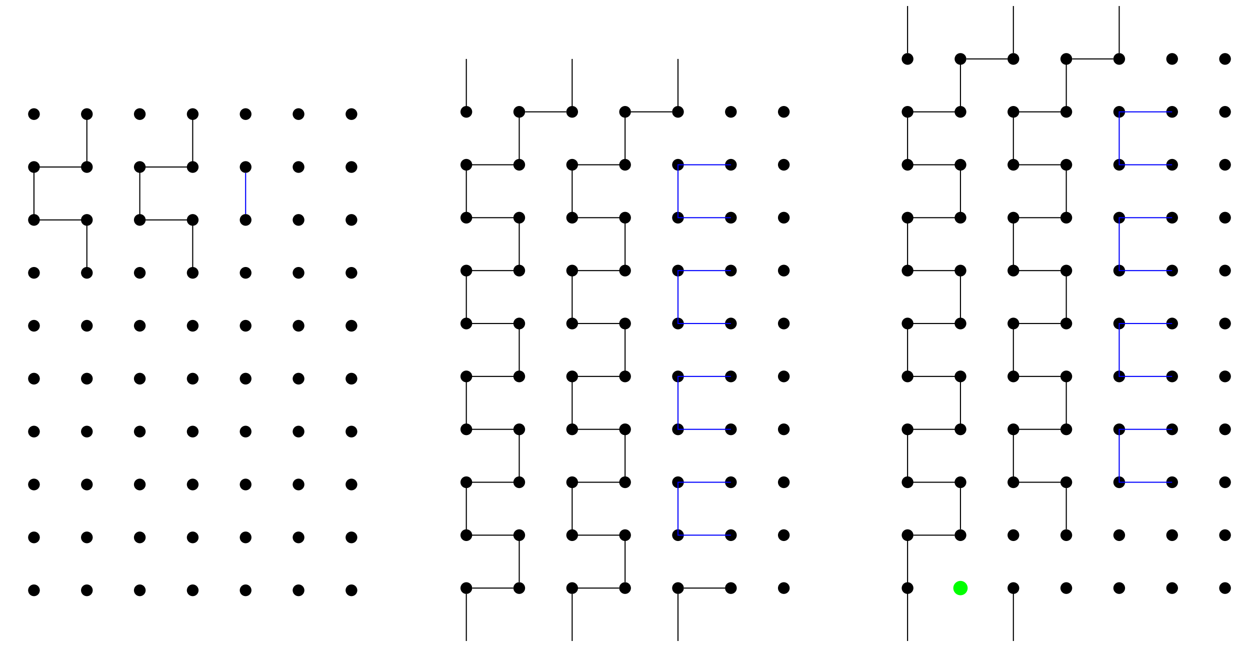


At this point, we have almost exactly specified all the edges in $P_i \cup C_i$ in the first b columns in the case where r is even. To complete the picture, observe that the vertex $(r-1, 0)$ must have degree 2. It cannot be connected to its top neighbor $(r-2, 0)$, because $(r-2, 0)$ already has degree 2. Thus, it is connected to its right neighbor, $(r-1, 1)$. Similarly, the vertex $(r-1, 2)$ has degree 2, so it must be connected to $(r-1, 3)$. Repeating this argument, we get that for all $j \in \{0, 2, \ldots, b\}$, the vertex $(r-1, j)$ (red in the diagram to the right), has an edge to $(r-1, j+1)$. These edges are all in P_i. But now, the horizontal edge $\{(r-1, b), (r-1, b+1)\}$ is in P_i and is adjacent to the horizontal edge $\{(r-2, b), (r-2, b+1)\}$ in C_i. This is a contradiction.



The same kind of argument works if r is odd. If r is odd, the resulting structure is depicted on the diagram to the right. To be precise, for all $l \in \{1, 3, 5, \ldots, r-4\}$, the vertices $(l, b+1), (l, b), (l+1, b), (l+1, b+1)$ are consecutive in C_i. (These are the blue paths of length 3 in the diagram.) Also, for all $l \in \{1, 3, \ldots, r-4\}$ and $j \in \{1, 3, \ldots, b-1\}$, the vertices $(l-1, j), (l, j), (l, j-1), (l+1, j-1), (l+1, j), (l+2, j)$ are consecutive in P_i. (These form the zig-zagging subpaths of P_i going downward.) Consider the vertex $(r-2, 0)$ (red in the diagram). Since it has degree 2, it must have an edge to the right to $(r-2, 1)$ and an edge down to $(r-1, 0)$. Now consider the vertex $(r-1, 1)$ (green in the diagram). It must have degree 2, but the only neighbor of the green vertex that does not have degree 2 yet is $(r-1, 2)$ to its right. Hence $(r-1, 1)$ can have degree at most 1, which is a contradiction. $\square$





(a) The picture of (P_i, C_i). The black edges are in P_i and the blue edge is the leftmost, topmost vertical edge of C_i.

(b) The forced connections if r is even. Blue edges are in the cycle, black edges are in the path. At the bottom right, a blue horizontal edge is adjacent to a black horizontal edge, which is a contradiction.

(c) The forced connections if r is odd. Blue edges are in the cycle, black edges are in the path. At the bottom left, the green vertex must have degree 2, but only has one available neighbor to connect to.

Figure 20: An illustration of the proof of Claim 6.8

6.5 Deriving the final contradiction

Together, Claims 6.7 and 6.8 show that the only remaining case is for the one where all the cycles $C_1, \dots, C_k$ are 2-cycles. We complete the proof of Theorem 4.1 by showing that this case is not possible in a minimal counterexample.

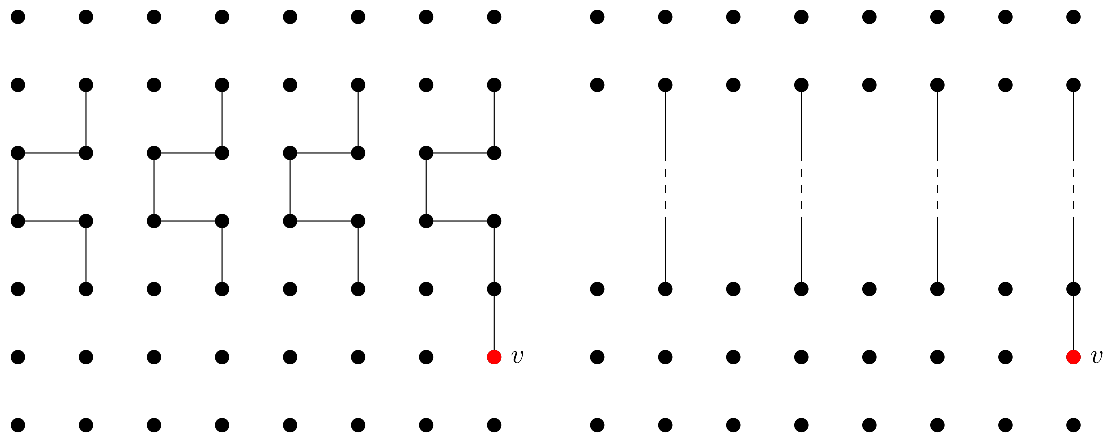
Claim 6.9. *In a minimal counterexample, it is impossible for all the cycles $C_1, \dots, C_k$ to be 2-cycles.*

Proof. If we are in the case where cycles $C_1, \dots, C_k$ are all 2-cycles, the picture must look like Figure 21. Note that, in particular, there are an even number of columns.

To complete the proof, we will show that there is a GG path T from 0 to v with cost at most the cost of P^*. To do this, we will use the minimal counterexample assumption to find a GG path T' in the cylinder graph with two rows deleted. We will then show that we can obtain our desired path T from T'.

Suppose rows i and $i+1$ are the two rows that contain the 2-cycles. Consider the graph G' obtained by deleting those two rows. Observe that we can turn P^* into a Hamiltonian path P' in G' by doing the following:

- Each time path P visits the two deleted rows, it must be via a subpath of the form $x, x+a_1, x+a_1-a_2, x+2a_1-a_2, x+2a_1, x+3a_1$ (possibly in the reverse order.)
- To obtain P' from P^*, replace all occurrences of the above subpath with the single vertical edge $\{x, x+3a_1\}$ (see Figure 21b).



(a) The picture of P^* in the case where every cycle is a 2-cycle.
(b) Reconnecting to a Hamiltonian path in the graph after deleting the two rows.

Figure 21: An illustration of the proof of Claim 6.9.

Recall that we defined $v := (v_1, c-1)$ to be the endpoint of P^* in the last column of G. Similarly, define $v' := (v_1', c-1)$ to be the endpoint of P' in the last column of G'. Then v_1' is either v_1 or $v_1 - 2$: It is v_1 if $v_1 < i$, and it is $v_1 - 2$ if $v_1 \geq i$.

Moreover, P' uses exactly $2g_1$ fewer horizontal edges than P. By minimality of our counterexample, we know the cheapest $0 - v'$ Hamiltonian path in G' is attained by a GG path. Hence, in G', there exists a GG path T' from 0 to v' with cost at most the cost of P'.

Let T be the cheapest 0-v GG path in G. For a path S, let $c(S)$ denote its cost (which in our case is the number of horizontal edges it uses). Then, by Lemma 6.10 below, we have that $c(T) \leq c(T') + 2$. Since $c(T') \leq c(P') = c(P) - 2g_1$, (and $g_1 \geq 1$), it follows that $c(T) \leq c(P)$, as claimed. □

Lemma 6.10. *If $x \in A_{r,c,m}$, then both x and $x+2$ are in $A_{r+2,c,m+1}$.*

Proof. Recall that $A_{r,c,m} = \{c + 2m - 2i \mod r : 0 \leq i \leq c + 2m\}$. It is useful to think of $A_{r,c,m}$ as $\{-(c+2m), -(c+2m)+2, \ldots, c+2m-2, c+2m\}$. That is, $A_{r,c,m}$ is the set of numbers between $-(c+2m)$ and $(c+2m)$ inclusive, with the same parity as $c+2m$, taken mod r.

Suppose $x = c + 2m - 2i \mod r$, where $0 \leq i \leq c + 2m$. By property 4 of Proposition 6.1, we may assume that $c + 2m < r$. Then

$$x = \begin{cases} c + 2m - 2i, & \text{if } 0 \leq c + 2m - 2i < r, \\ c + 2m - 2i + r, & \text{if } -r < c + 2m - 2i < 0. \end{cases}$$

We will now consider the two cases, and show that in each case, we have that both x and $x+2$ are in $A_{r+2,c,m+1}$. Before moving on, recall that $A_{r+2,c,m+1}$ is the following set:

$$A_{r+2,c,m+1} = \{c + (2m+2) - 2i \mod r+2 :\ 0 \leq i \leq c + 2m + 2\}.$$

<u>Case 1:</u> $0 \leq c + 2m - 2i < r$. Then we can write

$$x = c + 2m - 2i = c + (2m+2) - 2(i+1) \in A_{r+2,c,m+1},$$

and similarly

$$x + 2 = c + 2m - 2i + 2 = c + (2m+2) - 2i \in A_{r+2,c,m+1}.$$

Case 2: $-r < c + 2m - 2i < 0$. In this case, we write

$$\begin{aligned} x &= c + 2m - 2i + r \\ &= c + 2m + 2 - 2(i+2) + (r+2) \\ &= c + (2m+2) - 2(i+2) \mod (r+2) \\ &\in A_{r+2,c,m+1} \end{aligned}$$

and similarly

$$\begin{aligned} x + 2 &= c + 2m - 2i + r + 2 \\ &= c + (2m+2) - 2(i+1) + (r+2) \\ &= c + (2m+2) - 2(i+1) \mod (r+2) \\ &\in A_{r+2,c,m+1} \end{aligned}$$

Thus, in all cases, we have that both x and $x+2$ are in $A_{r+2,c,m+1}$, which completes the proof. □

7 Conclusion

Circulant edge costs present an intriguing special case of the TSP. Despite the substantial structure and symmetry, and despite being frequently stated as an open question, remarkably little has been known about the complexity of circulant TSP. By giving a polynomial-time algorithm for the two-stripe symmetric circulant TSP, we provide the first non-trivial complexity result.

The natural next question is to consider general symmetric circulant TSP, where all the edge lengths can potentially have finite cost. For that problem, a 2-approximation algorithm is known, but it is open whether it is polynomial-time solvable. As an intermediate step, one might consider a variant with some constant number of edge lengths having finite costs. One might wonder, e.g., if there is a dynamic programming approach that extends work from this paper to the constant-stripe case.

We also note that there are many open polyhedral questions for circulant TSP. Circulant TSP is one of the few non-trivial settings where the integrality gap of the subtour LP is exactly known (its integrality gap is 2; see Gutekunst and Williamson [12]). Gutekunst and Williamson [13] prove a facet-defining inequality using circulant symmetry, motivated by a class of two-stripe instances where the subtour LP's integrality gap is arbitrarily close to 2. One might wonder is there is an exact LP formulation for circulant TSP (in general or with a constant number of stripes), or as an intermediate step, a strengthening of the subtour LP.

A Proof of Proposition 3.10

Proposition 3.10. *Let P be a Hamiltonian path in an r by c cylinder graph, starting at $(0,0)$ and ending at $(x, c-1)$ for some row x. Suppose that P uses $2m+1$ horizontal edges between the first pair of columns, and then 1 horizontal edge between every subsequent pair of columns. Then $x \in A_{r,c,m}$.*

Proof. If P is a GG path, $x \in A_{r,c,m}$ definitionally. Because P uses 1 horizontal edge between every pair of columns after the first, it is only within the first two columns that it could have a non-GG path structure.

Claim A.1. *If $m = 0$, P is a GG path.*

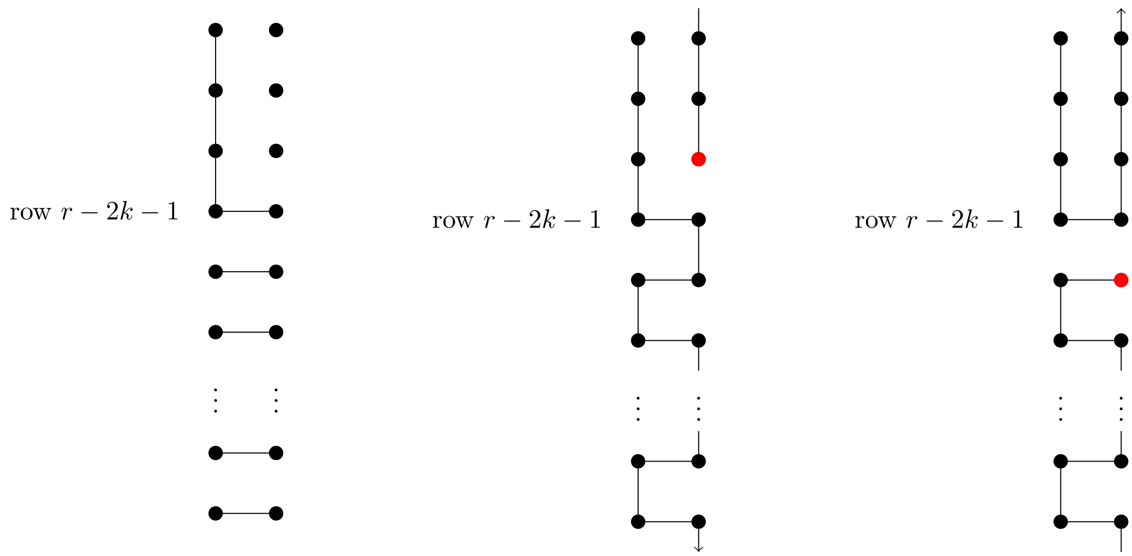


Figure 22: Argument that any Hamiltonian Path P in Proposition 3.10 starting with $r-2k-1$ vertical edges is either a GG path or can be replaced by a GG path with fewer edges. The leftmost image is for a generic path, including the vertical edges and forced horizontal edges between the first two columns. The middle and right image show the two possible ways to complete P's path through the first two columns.

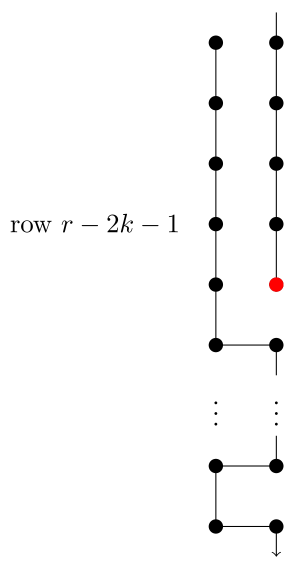


Figure 23: Replacing the rightmost path in Figure 22 with a GG path using two fewer horizontal edges.

Proof. First consider the case when $m = 0$, the path P must visit every vertex in the first column before moving to the second column. Since P starts at $(0,0)$, it either starts going vertically down (from $(0,0)$ to $(1,0)$) or wrapping up (from $(0,0)$ to $(r-1,0)$). In either case, it must be one of the tour types shown in Figure 9, i.e., a GG path. $\square$

If $m \geq 1$, there is at least one extra pair of horizontal edges between the first two columns. We proceed by considering the first horizontal edge.

Claim A.2. *If $m \geq 1$ and the first $r-2k-1$ edges of P are vertical, $x \in A_{r,c,m}$.*

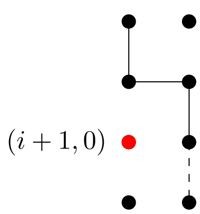
Proof. Suppose that the first $r-2k-1$ edges of P are vertical. By cylindrical symmetry, it is without loss of generality to assume that P starts by following $(0,0),(0,1),...,(0,r-2k-1)$. To include $2m+1$ horizontal edges between the first two columns, we must have horizontal edges between $(0,i)$ and $(1,i)$ for $r-2k-1 \le i \le r-1$. See Figure 22. Note that there are two possible ways to complete P's path through the first two columns. If, from vertex $(1,r-2k-1)$, the path continues going down to $(1,r-2k)$ (in the same direction as the original edges $(0,0),(1,0),...,(r-2k-1,0)$), then P is a GG path (the middle image of Figure 22). Otherwise the path goes up from $(r-2k-1,1)$ to $(r-2k-2,1)$, and must be completed as in the right most image of Figure 22. The first two columns of such a path can be replaced, as in Figure 23, covering P into a GG path, using two fewer horizontal edges, so that $(x,c-1) \in A_{r,c,m-1} \subset A_{r,c,m}$. Formally, we delete the horizontal edges from $(r-2k-1,0)$ to $(r-2k-1,1)$ and from $(r-2k,0)$ to $(r-2k,1)$, and replace them with vertical edges from $(r-2k-1,1)$ to $(r-2k,1)$ and from $(r-2k-1,1)$ to $(r-2k,1)$. □

At this point, what remains are paths P that start at $(0,0)$ and where at least one of the first $r-2k-1$ edges is horizontal. We consider what happens if the first edge is vertical. As above, it is without loss of generality to assume that the first edge is from $(0,0)$ to $(1,0)$.

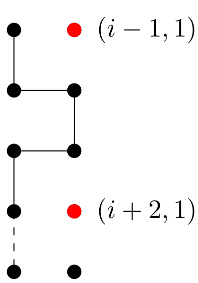
Suppose that the first horizontal edge is from $(i,0)$ to $(i,1)$ with $i \ge 1$, so that P contains the sequence $(i-1,0),(i,0),(i,1)$. Since P eventually returns to the first column (and uses one edge between the second and third columns), the next edge in this sequence is either $(i+1,1)$ or $(i-1,1)$.

Claim A.3. *Suppose that P contains the sequence $(i-1,0),(i,0),(i,1),(i+1,1)$. Then P must be a GG path and so $x \in A_{r,c,m}$.*

Proof. First, the sequence must continue $(i-1,0),(i,0),(i,1),(i+1,1),(i+1,0)$, as otherwise the vertex $(i+1,0)$ cannot possibly have degree two (see the red vertex below).



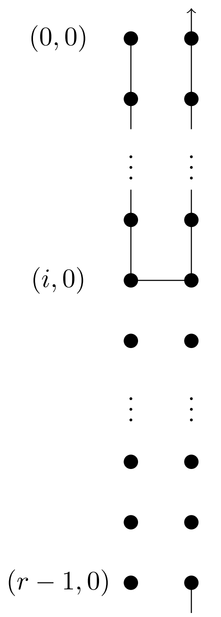
Analogously, P must continue alternating and proceed as $(i-1,0),(i,0),(i,1),(i+1,1),(i+1,0),(i+2,0)$. Otherwise the nodes $(i-1,1)$ and $(i+2,1)$ cannot both have degree two (see the red nodes below; note that $(i-1,0)$ is fully saturated: either $i=1$ and $(0,0)$ has degree 1 in a Hamiltonian path, or $i>1$ and our assumption that $(i,0)$ to $(i,1)$ is the first horizontal edge in P implies that $(i-1,0)$ is also adjacent to $(i-2,0)$ and has so $(i-1,0)$ has degree 2.



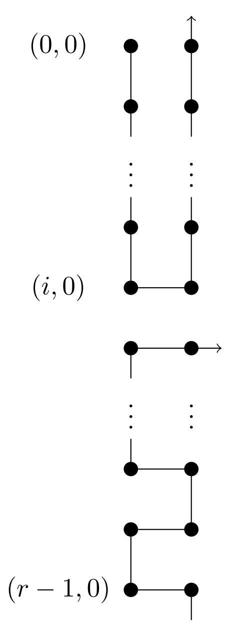
These arguments continue iteratively, showing that P must alternate horizontal and vertical edges, proceeding until all $2m+1$ horizontal edges have been used. Since P is Hamiltonian and visits every vertex in the first column, we must have that P is a GG path of the form where the first edge goes from $(0,0)$ to $(1,0)$. □

Claim A.4. *Suppose that P contains the sequence $(i-1,0),(i,0),(i,1),(i-1,1)$. Then $x \in A_{r,c,m}$.*

Proof. This case proceeds largely as in Claim A.3. First we note that P must continue as $(0,0),(1,0),...,(i-1,0)$, $(i,0),(i,1),(i-1,1),(i-2,1),...,(1,0),(r-1,1)$



We can see that P must continue from $(r-1,1)$ to $(r-1,0)$ (so that $(r-1,0)$ has degree 2) and then to to $(r-2,0)$. As in Claim A.3, at $(r-20)$, P must continue to $(r-2,1)$, and continue alternating between the first and second column. Ultimately P must proceed as below. Since P uses exactly $2m+1$ horizontal edges between the first two columns, the horizontal edges must be in in the last $2m+1$ rows: $r-1, r-2, r-3, ..., r-2m-1$. P leaves the second column at the second horizontal edge, at row $r-2m$. Note that we can replace the part of P in the first two columns using a GG path with two fewer edges: delete the edges from $(i,0)$ to $(i,1)$ and from $(i+1,0)$ to $(i+1,1)$, and replace them with edges from $(i,0)$ to $(i+1,0)$ and from $(i,1)$ to $(i+1,1)$.



□

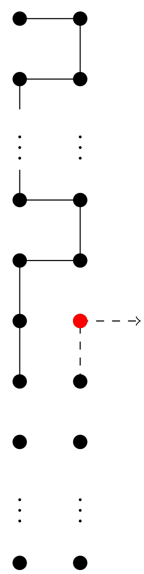
The only remaining case we must consider is if the first edge of P is horizontal. Again appealing to cylindrical symmetry, we assume that P starts $(0,0),(0,1),(1,1)$.

Claim A.5. *Suppose that P contains the sequence $(0,0),(0,1),(1,1)$. Then $x \in A_{r,c,m}$.*

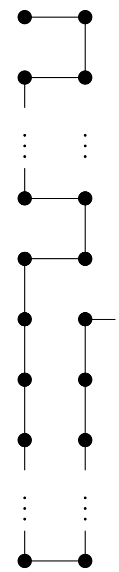
Proof. As in our previous claims, we trace out what must happen with P. We note that P must continue from $(1,1)$ to $(1,0)$ to $(2,0)$ (as otherwise $(1,0)$ cannot have degree 2). Similarly, if P proceeds from $(2,0)$ to $(2,1)$, it must continue $(3,1)$ to $(3,0)$ to $(4,0)$. More generally, P will alternate between the first and second column until either 1) P visits every vertex in the first column or 2) P contains at least two sequential vertical edges in the first column. The former case can only happen if $r = 2m+1$, in which case P is definitionally a GG path, and so we restrict our attention to the latter case.

In this case, P alternates between the first and second column until it reaches some sequence $(i-1,0),(i,0),(i+1,0)$ as shown below. Consider the vertex $(i,1)$, indicated in red below. This vertex must have degree two, and so P must contain the sequence $(i-1,1),(i,1),(i,2)$, indicated with dashed lines.

$(i,0)$



Provided $i+1 \neq r-1$, there are more vertices to visit in the first and second column. Hence P cannot contain the edge from $(i-1,0)$ to $(i-1,1)$. Since both $(i-1,0)$ and $(i-1,1)$ must have degree 2, P must contain edges from $(i-1,0)$ to $(i-2,0)$ and from $(i-1,1)$ to $(i-2,1)$. Continuing this argument, we see that P must contain sequential pairs of vertical edges in the first and second column, so that P must proceed $(i,0),(i-1,0),(i-2,0),...,(r-1,0),(r-1,1),(r-2,1),(r-3,1),...,(i+1,1),(i,1)$ as indicated below.



Note that we can replace the part of P in the first two columns using a GG path with two fewer edges: delete the edges from $(0, 0)$ to $(0, 1)$ and from $(r-1, 0)$ to $(r-1, 1)$, and replace them with edges from $(0, 0)$ to $(r-1, 0)$ and from $(0, 1)$ to $(r-1, 1)$. □

The above claims complete our proof. □

B Confirming the Conjecture of Gerace and Greco [8, 10]

Recall our main result

Theorem 2.6. *Let $r = \frac{n}{g_1}$ and $c = g_1$. Suppose the cylindrical coordinates of $-a_2$ are $(x, c-1)$. Let m^* be the smallest integer value of m such that $m \geq -\frac{c}{2}$, and $x \in \{2m+c, -(2m+c)\} \mod r$. Then:*

- *If $m^* \leq 0$, the cost of the optimal tour is c.*
- *If $0 < 2m^* < c-2$, the cost of the optimal tour is $c + 2m^*$.*
- *If $2m^* \geq c-2$ or m^* does not exist, the cost of the optimal tour is $2c-2$.*

We show that this result resolves Greco and Gerace's [8, 10] conjecture. Greco and Gerace defined

$$S = \left\{ y : 0 \leq y < \frac{n}{g_1},\ (2y - g_1)a_1 + g_1 a_2 \equiv_n 0 \right\}.$$

Their main result (Theorem 4.4 in Gerace and Greco [8]) states:

1. If S is empty, the cost of the optimal tour is $2c-2$.
2. Let $y_1 = \min\{S\}$ and $y_2 = \max\{S\}$. If $y_1 \leq g_1$, the cost of the optimal tour is c
3. Otherwise, let $m = \min\{y_1 - g_1, \frac{n}{g_1} - y_2\}$. There exists a tour of cost $c + 2m$

Greco and Gerace [8, 10] conjectured that, in cases where S is nonempty, either the tours in case 1 or 3 above are optimal. That is, for cases where S is non empty, the cost of the optimal tour is $\min\{c+2m, 2c-2\}$.

We now prove that our characterization confirms this conjecture. To prove this conjecture, we recall from Proposition 2.9 and Theorem 2.6 that the lower bound c is optimal if and only if there exists an integer y, with $0 \leq y \leq g_1$, such that

$$(2y - g_1)a_1 + g_1 a_2 \equiv_n 0.$$

Recall also that, if $\frac{n}{g_1} - 1 \leq g_1$, then S is either empty or $\min S \leq \frac{n}{g_1} - 1 \leq g_1$ so that the lower bound is optimal.

To resolve Greco and Gerace's conjecture, we show the following:

Theorem B.1. *Consider an instance of two-stripe TSP where $g_1 + 1 < \frac{n}{g_1}$. Let*

$$S = \left\{ y : 0 \leq y < \frac{n}{g_1},\ (2y - g_1)a_1 + g_1 a_2 \equiv_n 0 \right\}.$$

Assume that S is nonempty, that $y_1 = \min\{S\}$ and $y_2 = \max\{S\}$, and that $y_1 > g_1$. Finally, assume that the cost of the optimal solution to the two stripe instance is strictly between c and $2c-2$. Then the cost of the optimal solution to the two-stripe instance is $c + 2m^*$ if and only if $m^* = \min\{y_1 - g_1, \frac{n}{g_1} - y_2\}$.

Proof.

Claim B.2. *Suppose that the optimal solution to the two-stripe is $c+2m^*$ with $c < c+2m^* < 2c-2$. Then $\min\{y_1 - g_1, \frac{n}{g_1} - y_2\} \leq m^*$.*

Recall from Corollary 3.8 that, if the cost of the optimal solution is $c + 2m^*$ with $c < c + 2m^* < 2c - 2$ (i.e., if the optimal solution is a GG path plus a $(0 - (-a_2))$ edge), then the corresponding GG path ends in the last column at row $c + 2m^* \mod r$ or at row $-(c + 2m^*) \mod r$.

Case 1: The optimal tour corresponds to a GG path ending at row index $c + 2m^* \mod r$.

First, suppose that the optimal tour corresponds to a GG path ending at row index $c + 2m^*$. The corresponding GG path is then a path that starts going vertically up (see the proof of Theorem 5.2) and ending at $-a_2$. This path uses $g_1 - 1 + 2m^*$ edges of length $\pm a_2$ with $2m^*$ of those edges alternating between the first two columns. Since it uses $n-1$ total edges, and all other edges are vertically up (i.e. of length $-a_1$), we have that it uses:

- m^* edges of length $-a_2$
- $g_1 - 1 + m^*$ edges of length a_2
- $n - 1 - (g_1 - 1 + 2m^*) = n - g_1 - 2m^*$ edges of length $-a_1$.

Since it ends at $-a_2$, we know that

$$-a_2 \equiv_n -m^* a_2 + (g_1 - 1 + m^*)a_2 + (n - g_1 - 2m^*)(-a_1) \equiv_n (g_1 - 1)a_2 + (g_1 + 2m^*)a_1.$$

Rearranging, we see that

$$2m^* a_1 \equiv_n -g_1 a_2 - g_1 a_1.$$

Now let $y = m^* + g_1 \geq 0$ Then

$$\begin{aligned}(2y - g_1)a_1 + g_1 a_2 &= (2(m^* + g_1) - g_1)a_1 + g_1 a_2 \\ &= 2m^* a_1 + g_1 a + 1 + g_1 a_2 \\ &\equiv_n -g_1 a_2 - g_1 a_1 + g_1 a_1 + g_1 a_2 \\ &= 0\end{aligned}$$

Hence, $y \in S$ provided $y < \frac{n}{g_1}$. To see that $y < \frac{n}{g_1}$ suppose not. Let $y' = y - \frac{n}{g_1}$. By assumption, $y' \geq 0$. Moreover

$$y' = y - \frac{n}{g_1} = m^* + g_1 - \frac{n}{g_1} < m^*,$$

since we assume $g_1 + 1 < \frac{n}{g_1}$. Moreover, since we assume $c + 2m^* < 2c - 2$, we have that $m^* < c = g$. Thus $0 \leq y' < \frac{n}{g_1}$. Finally,

$$y'a_1 = (y - \frac{n}{g_1})a_1 \equiv_n ya_1,$$

so that $y' \in S$ and $\min\{S\} < g_1$. Then, however, the optimal solution must cost c (by, e.g., Proposition 2.12). Thus we cannot have $y \geq \frac{n}{g_1}$ and must have $y \in S$. Since $y_1 = \min S$, $y_1 \leq y$. Finally,

$$\min\{y_1 - g_1, \frac{n}{g_1} - y_2\} \leq y_1 - g_1 \leq y - g_1 = m^*.$$

Case 2: The optimal tour corresponds to a GG path ending at row index $-(c + 2m^*) \mod r$.

This case proceeds almost identically to Case 1. If the GG path ends at row index $-(c + 2m^*) \mod r$, then the only difference is that it uses $(n - g_1 - 2m^*)$ edges of length a_1. Since it ends at $-a_2$, we know that

$$-a_2 \equiv_n -m^*a_2 + (g_1 - 1 + m^*)a_2 + (n - g_1 - 2m^*)(a_1) \equiv_n (g_1 - 1)a_2 - (g_1 + 2m^*)a_1.$$

Rearranging, we see that

$$-2m^*a_1 \equiv_n -g_1a_2 + g_1a_1.$$

Now let $y = \frac{n}{g_1} - m^*$. Then

$$\begin{aligned}(2y - g_1)a_1 + g_1a_2 &= (2(\frac{n}{g_1} - m^*) - g_1)a_1 + g_1a_2 \\ &\equiv_n -2m^*a_1 - g_1a_1 + g_1a_2 \\ &\equiv_n -g_1a_2 + g_1a_1 - g_1a_1 + g_1a_2 \\ &= 0\end{aligned}$$

Moreover $y < \frac{n}{g_1}$ since $m^* \geq 1$. Since $c + 2m^* < 2c - 2$ we again have that $m^* < c = g_1$ and, further, since $g_1 < \frac{n}{g_1}$, we have that $m^* < \frac{n}{g_1}$. Thus $0 \leq y < \frac{n}{g_1}$ and again $y \in S$. Since $y_2 = \max S$, $y_2 \geq y$. Finally,

$$\min\{y_1 - g_1, \frac{n}{g_1} - y_2\} \leq \frac{n}{g_1} - y_2 \leq \frac{n}{g_1} - y = m^*.$$

These two cases resolve Claim B.2. To finish proving the theorem, we want to show the following

Claim B.3. *Suppose that $m = \min\{y_1 - g_1, \frac{n}{g_1} - y_2\}$ with $m > 0$. Then the cost of the optimal two-stripe solution is at most $c + 2m$.*

To prove this claim, we need only construct a Hamiltonian tour of cost $c + 2m$. This is done in Theorem 4.4 of Gerace and Greco [8] using GG paths. For completeness, we sketch the proof here. We have two cases that parallel the previous two cases.

Case 1: $m = y_1 - g_1$. Suppose that $m = y_1 - g_1$. Consider a GG path where all edges are vertically up (of length $-a_1$) of cost $(c-1) + 2m$. This is, by construction, a Hamiltonian path. It gives rise to a Hamiltonian tour of cost $c + 2m$ provided it ends at $-a_2$. As in Case 1 of Claim B.2, we can calculate that it ends at

$$(g_1 - 1)a_2 + (g_1 + 2m)a_1 \mod n.$$

Since $y_1 \in S$, we have that

$$g_1 a_2 + (2y_1 - g_1)a_1 \equiv_n 0.$$

Using the fact that $m = y_1 - g_1$, we get:

$$\begin{aligned}(g_1 - 1)a_2 + (g_1 + 2m)a_1 &= (g_1 - 1)a_2 + g_1 a_1 + 2ma_1 \\ &= (g_1 - 1)a_2 + g_1 a_1 + 2(y_1 - g_1)a_1 \\ &= (g_1 - 1)a_2 + (2y_1 - g_1)a_1 \\ &= -a_2 + g_1 a_2 + (2y_1 - g_1)a_1 \\ &\equiv_n -a_2,\end{aligned}$$

using the fact that $g_1 a_2 + (2y_1 - g_1)a_1 \equiv_n 0$ in the last line. Thus this GG path gives rise to a Hamiltonian tour of cost $c + 2m$.

Case 1: $m = y_1 - \frac{n}{g_1} - y_2$. Suppose that $m = \frac{n}{g_1} - y_2$. Consider a GG path where all edges are vertically down (of length a_1) of cost $(c - 1) + 2m$. This is, by construction, a Hamiltonian path. It gives rise to a Hamiltonian tour of cost $c + 2m$ provided it ends at $-a_2$. As in Case 2 of Claim B.2, we can calculate that it ends at

$$(g_1 - 1)a_2 - (g_1 + 2m)a_1 \mod n.$$

Since $y_2 \in S$, we have that

$$g_1 a_2 + (2y_2 - g_1)a_1 \equiv_n 0.$$

Using the fact that $m = \frac{n}{g_1} - y_2$, we get:

$$\begin{aligned}(g_1 - 1)a_2 - (g_1 + 2m)a_1 &= (g_1 - 1)a_2 - (g_1 + 2\frac{n}{g_1} - 2y_2)a_1 \\ &\equiv_n (g_1 - 1)a_2 + (2y_2 - g_1)a_1 \\ &= -a_2 + g_1 a_2 + (2y_2 - g_1)a_1 \qquad \equiv_n -a_2,\end{aligned}$$

using the fact that $g_1 a_2 + (2y_2 - g_1)a_1 \equiv_n 0$. in the last line. Thus this GG path gives rise to a Hamiltonian tour of cost $c + 2m$.

□

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